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CARROLLIAN LIMITS OF MODMAX

A Lie point symmetry analysis

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Last but not least, thank you, dear reader, for taking the time for reading this. I hope you enjoy the ride.

Resumen

Los límites Carrollianos de teorías de campos Lorentzianas se han encontrado recientemente estudiadas, en los últimos años, como resultado de un renovado interés en teorías y geometría Carrolliana desde el lado de física teórica. De ahí nace la decisión de estudiar los límites Carrollianos de la teoría de Maxwell Modificado (ModMax, por su escritura en inglés), que es la única extensión no-linear de la teoría de Maxwuell con invarianza conforme y de dualidad en el vacío. El presente trabajo contiene tanto una derivación de ambos límites a nivel de las ecuaciones de movimiento como una construcción de una formulación Hamiltoniana para cada uno. Se encontró que el límite magnético tiene una contribución no-linear no-nula a las ecuaciones de movimiento controlada por el parámetro de ModMax γ y que esta admite una biyección con el límite Carrolliano magnético de la teoría de Maxwell. Cabe destacar que estos límites no son equivalentes pues existen configuraciones que son solución de uno de ellos y no así del otro. En particular, existen soluciones que muestran una dependencia explícita del parámetro de ModMax γ. Se encontró que el límite Carrolliano eléctrico de ModMax es equivalente al de Maxwell, siendo carente de contribución no-lineal.

Las simetrías de los límites Carrollianos de Maxwell fueron obtenidas empleando el método de simetrías puntuales de Lie y se probó que constituyen también simetrías de sus correspondientes contrapartes en los límites Carrollianos de ModMax mediante la anteriormente mencionada biyección. Estas simetrías incluyen desplazamientos finitos tanto temporales como espaciales, rotaciones espaciales, impulsos Carrollianos, dilaciones temporales, dilaciones espaciales, transformaciones conformes especiales Carrollianas de nivel $k = 2$, súper-traslaciones temporales, dilataciones de campo y una simetría interna que surge como legado de la simetría de dualidad en la versión Lorentziana. Debido a la separación de las dilaciones espacio-temporales en dilaciones espaciales y temporales, estas simetrías no caben dentro de ninguna clasificación de grupos conformes Carrollianos. Sin embargo, al tomar el sub-álgebra diagonal se encontró que esta satisface los criterios necesarios para pertenecer al álgebra conforme Carrolliana de nivel 2.

Keywords – Carroll, Carrollian limits, Electrodynamics, Conformal, Non-linear, Lie point symmetries

Abstract

Carrollian limits of Lorentzian field theories have recently found themselves studied in the last few years as a result of renewed interest on Carrollian theories and geometry on the theoretical part of physics. Thus the decision to study the Carrollian limits of Modified-Maxwell (ModMax) theory, the unique conformal and duality invariant non-linear extension of Maxwell theory, was taken. The present work contains a derivation of these limits at the level of the equations of motion and the construction of a Hamiltonian formulation of them. It was found that the magnetic limit has a non-vanishing non-linear contribution to the equations of motion controlled by the ModMax parameter γ and that it admits a bijection with the magnetic limit of Maxwell theory, however, these two are not equivalent since there exists solutions on one side that are not solutions of the other and, in particular, there exists solutions with explicit dependence on the ModMax parameter γ. The electric limit of ModMax was found to be equivalent of that of Maxwell theory, having no non-linear contribution.

The symmetries of the Carrollian limits of Maxwell theory were obtained through the use of Lie point symmetry method and are proven to also be symmetries of the Carrollian limits of ModMax theory by use of the Maxwell-ModMax bijection. These symmetries include time translations, space translations, Carrollian boosts, spatial rotations, time dilations, space dilations, special conformal transformations, field dilations, super-translations on the temporal part and an internal symmetry that corresponds to a legacy of duality invariance of the Lorentzian theory. Because of the separation of the space-time dilation into space and time dilations, the resulting algebra of symmetries does not belong in any categorization of conformal Carrollian algebras, however, by taking the diagonal sub-algebra it was found these belong to a subset of the conformal Carrollian algebra of level 2.

Keywords – Carroll, Carrollian limits, Electrodynamics, Conformal, Non-linear, Lie point symmetries

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Chapter 1

Introducción

1.1 Sobre la estructura

Citando a Tolkien, *no siempre resplandece lo que es oro* Tolkien (1954) y, en este caso, no siempre es inmediatamente útil todo lo que está escrito en este trabajo para entender los resultados presentados.

El capítulo 3 comienza con una presentación básica de geometría pseudo-Riemanniana, que puede ser saltada si el lector ya se encuentra familiarizado con el tópico. Esto se encuentra allí en caso de que existan dudas sobre las convenciones usadas. Luego hay una exposición sobre el grupo de Lorentz y se sugiere que el lector la mantenga presente con objeto de comparar con las simetrías encontradas en la sección destinada a Carroll.

Los siguientes capítulos están compuestos por definiciones útiles e importantes de geometría Carrolliana además de un breve ejemplo compuesto por el límite Carrolliano del campo escalar libre, donde se empleó el método de simetrías de contacto de Lie por primera vez en este trabajo.

El capítulo 6 contiene una breve revisión de la teoría de Maxwell, incluyendo la derivación de sus simetrías y su formulación Lagrangiana y Hamiltoniana.

El capítulo 7 es quizá la sección más relevante de esta tesis, debido a que contiene tanto la derivación de las simetrías que aparecen también en los límites de ModMax, así como un análisis de a qué corresponden.

El capítulo 9 es la culminación de este trabajo, pues es donde se demuestra que las simetrías encontradas previamente en el capítulo 6 corresponden también a simetrías de estos límites. La formulación Hamiltoniana de estos se construye también aquí.

Se incluye también un apéndice que contiene lo básico del método de simetrías de contacto de Lie y al lector se le recomienda leerlo si busca poseer un mejor entendimiento de este trabajo.

1.2 Estado del arte

El grupo de Carroll fue encontrado por primera vez por Levi-Leblond y Bacri en el artículo Bacry and Levy-Leblond (1968), donde fueron clasificados todos los posibles grupos cinemáticos. Los requerimientos considerados para ser considerado un grupo cinemático son poseer una noción de causalidad, poseer espacio isotrópico, admitir paridad y reversión temporal, y que las transformaciones inerciales compongan un subgrupo no compacto del grupo total de transformaciones, lo que corresponde a pedir que existan boosts. Mientras que el grupo de Poincaré posee las transformaciones de Lorentz, que corresponden a rotaciones hiperbólicas que mezclan espacio y tiempo, su equivalente en Carroll transforma solo la coordenada temporal, dejando la parte espacial invariante. Las traslaciones finitas temporales y espaciales son también parte del grupo de Carroll.

En este artículo también se encontró que el grupo de Galileo, responsable de las transformaciones de simetría en mecánica clásica, es también uno de los posibles grupos cinemáticos. Notoriamente, en los últimos años, se ha mostrado que el grupo de Carroll admite una noción de dualidad con este grupo, véase Figueroa-O'Farrill (2022), que ambos pueden ser tratados de forma unificada como casos particulares de una variedad de Bargmann extendida Duval et al. (2014c) y que existe un método para construir teorías invariantes de Galileo a partir de teorías invariantes de Carroll y vice-versa utilizando Lagrangianos semilla en Bergshoeff et al. (2023c).

El grupo de Carroll fue obtenido como una contracción del álgebra de Poincaré al considerar un caso límite en el que la velocidad de la luz se acerca a cero, lo que ha probado tener relevancia física con relación a superficies nulas Herfray (2022) y su extensión conforme siendo isomorfa al grupo de Bondi Metzer Sachs Duval et al. (2014b). En este contexto, aspectos Carrollianos han comenzado a ser explorados con relación a holografía de espacio plano Bagchi et al. (2023b) y como modelos efectivos sobre superficies nulas en variedades Lorentzianas que admiten movimiento aparente Marsot (2023). El movimiento en geometrías de Carroll se pensaba en un principio imposible debido a la clara separación causal de dicha geometría pero se han encontrado modelos que poseen propagación fuera de las lineas de luz[1] Ecker et al. (2024) y ha sido mostrado que particulas Carrollianas acopladas admiten dinámica no-trivial Bergshoeff et al. (2014). Otra posible aplicación fue encontrada en Bagchi et al. (2024b), donde el flujo de Gubser, que provee de un modelo analítico para describir la dinámica de espacio-tiempo de un plasma de gluónes y quarks producidos en colisiones de iones pesados, junto a sus suposiciones de simetría asociadas, es argumentado surgen naturalmente como una consecuencia de las simetrías Carrollianas de un fluido Carrolliano. Dado el contexto de que teorías Carrollianas admiten dinámicas no-triviales, los límites Carrollianos de p-formas, incluyendo el caso del campo escalar y el de teorías de Yang-Mills, fueron estudiadas en Henneaux and Salgado-Rebolledo (2021).

Las representaciones unitarias irreducibles de un dipolo de Carroll fueron encontradas y clasificadas en Figueroa-O'Farrill et al. (2023b) como una continuación del trabajo presentado en Figueroa-O'Farrill et al. (2023a), donde la correspondencia Carroll-fractón (que son particulas que no pueden moverse) fue establecida. Las G-estructuras de Carroll fueron clasificadas en término de sus torsiones intrínsecas en Figueroa-O'Farrill (2020).

Desde el lado de gravitación, el límite Carrolliano de la acción de Einstein-Hilbert fue obtenida en Guerrieri and Sobreiro (2021). Las simetrías asintóticas para teorías gravitacionales Carrollianas en $(3+1)$ dimensiones, obtenidas de contracciones ultra-relativistas $(c \to 0)$ eléctricas y magnéticas de relatividad general fueron analizadas en Pérez (2021). Un principio de acción a la Cartan, a primer orden, invariante bajo el grupo homogéneo de Carroll para gravedad Carrolliana eléctrica fue presentada en Pekar et al. (2024). Ha sido sugerido en de Boer et al. (2022) que las simetrías de Carroll podrían ser relevantes para energía oscura e inflación. Es más, en el artículo Najafizadeh (2024a) se sugiere que quizá partículas Carrollianas sean un candidato a materia oscura debido a que generan un campo gravitacional que apunta hacia fuera.

El artículo original Bacry and Levy-Leblond (1968) ha recibido un aumento

[1]En contraposición con los conos de luz usuales.

significativo en su número de citas en la última década y una revisión exhaustiva de todos los trabajos relevantes va más allá del objetivo de este trabajo. Sin embargo, una buena revisión del tópico puede ser encontrada en Bergshoeff et al. (2023a) o Ecker et al. (2024) y las referencias que estos contienen.

Chapter 2

Introduction

2.1 On the structure

To quote Tolkien, *all that is gold does not glitter* Tolkien (1954), and in this case not all that is written in this document is immediately useful for understanding the results presented.

Chapter 1 starts with a presentation of basic stuff of pseudo-Riemannian geometry, which can be skipped if the reader is already familiar with the subject. This is in the place it is in case there are doubts on conventions used. Next in this chapter is an exposition on the Lorentz group and the reader is advised to keep its contents in mind to compare them with the symmetries obtained in the Carrollian part.

The next chapters are composed of important and useful definitions of Carrollian geometry as well as a brief example of the Carrollian limits of the free scalar field theory, where the Lie point symmetry method is employed for the first time in this work.

Chapter 6 contains a brief review of Maxwell theory, including a derivation of its symmetries, its Lagrangian description and its Hamiltonian description.

Chapter 7 is perhaps the most relevant part of this thesis, as it contains both the derivation of the symmetries that appear also in ModMax limits and an analysis of what they are.

Chapter 9 is the culmination of this work, where it is demonstrated that the symmetries found in the previous chapter are also symmetries of these limits. The

Hamiltonian formulation of the electric and magnetic limit of ModMax is also constructed here.

An appendix containing the basics of the Lie point symmetry method is also included in this document and the reader is advised to read it if a better insight on how the work was done is desired.

2.2 State of the art

The Carroll group was first found in the paper Bacry and Levy-Leblond (1968) by Levi-Leblond and Bacry in 1968. In this paper, all possible kinematic groups were classified. The requirements for being considered a possible kinematic group were having a notion of causality, having isotropy of space, admitting parity and time reversal and that inertial transformations form a non-compact subgroup of the total group of transformations, that is the requirement of having boosts as one of the symmetries. While the Poincaré group has boosts that are hyperbolic rotations that mix space and time, boosts in the Carroll group affect only the time coordinate while leaving the space part unaffected. Finite space and time translations and space rotations are also part of the Carroll group.

This paper also found the Galilean group, the symmetry group of non-relativistic mechanics, to be one of the possible kinematic groups. Notably enough in the last few years the Carroll group was shown to admit a notion of duality with the Galilean group Figueroa-O'Farrill (2022) and both of them can be treated in a unified manner as particular cases in an extended Bargmann manifold Duval et al. (2014c) and a method for constructing Carroll invariant theories from Galilei invariant ones was developed by the employment of seed Lagrangians in Bergshoeff et al. (2023c).

The Carroll group was obtained as a contraction of the Poincaré algebra by considering the limiting case of the speed of light going to zero, something that has been proven to have physical significance in relation with null surfaces Herfray (2022) and its conformal extension being isomorphic to the Bondi–Metzner–Sachs group Duval et al. (2014b). In this context, Carrollian aspects begun to be explored in relationship with flat space holography Bagchi et al. (2023b) and as effective models on null hyper-surfaces in a Lorentzian spacetime admitting apparent motion Marsot (2023). Motion in Carrollian geometries was first believed to not be possible due to the clearly separated causal structure of this geometry but it has been found that models with interactions admit propagation outside the light-line[1] Ecker et al. (2024) and it has been proved that coupled Carroll particles admit non-trivial dynamics Bergshoeff et al. (2014). Another possible application was found in Bagchi et al. (2024b), where Gubser flow, which provides an analytic model for describing the spacetime dynamics of the quark-gluon plasma produced in heavy-ion collisions, along with its associated symmetry assumptions are argued to arise naturally as a consequence of Carrollian symmetries for a conformal Carroll fluid. Given the context that Carrollian theories admit non-trivial dynamics, Carrollian limits of general p-forms were studied, including the scalar case and Yang-Mills theory in Henneaux and Salgado-Rebolledo (2021).

The Unitary Irreducible Representations (UIRs) of Carroll and dipole groups were found and classified in Figueroa-O'Farrill et al. (2023b) as a continuation of the work presented in Figueroa-O'Farrill et al. (2023a), where the Carroll-fracton (which are particles that cannot move) correspondence was established. Carrollian G-structures were classified in terms of their intrinsic torsion in Figueroa-O'Farrill (2020).

From the gravity side, the Carroll limit of the Einstein-Hilbert action was obtained in the year 2021 in the paper Guerrieri and Sobreiro (2021). Asymptotic symmetries in Carrollian gravitational theories in $(3+1)$-space-time dimensions obtained from magnetic and electric ultrarelativistic ($c \to 0$) contractions of General Relativity were analyzed in Pérez (2021). A Cartan-like first-order homogeneous-Carroll-invariant action principle for electric Carrollian gravity was presented in Pekar et al. (2024). It was suggested in a 2022 paper de Boer et al. (2022) that Carrollian symmetries might be relevant for dark energy and inflation, furthermore, a recent 2024 paper Najafizadeh (2024a) suggests that Carroll particles may be a candidate for dark matter as they generate an outward gravitational field.

The original paper Bacry and Levy-Leblond (1968) has received a boost in the number of citations during the last decade and a thorough revision of all relevant works is beyond the scope of this work. Nevertheless, a good overview of the subject can be found in Bergshoeff et al. (2023a) or Ecker et al. (2024) and the references therein.

[1] In contrast to the usual light-cone.

Part I

Preamble and definitions

Chapter 3

Poincare group, algebra and induced structure

“ *Education had been easy. Learning things had been harder.* ”

Hogfather, Terry Pratchett Pratchett (1996).

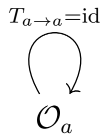
Relativity principles are a common theme of study in physics, mostly thanks to Albert Einstein’s work on general relativity but have existed for quite a long time as they arise from a very simple yet important question:

How do we compare measurements between observers?

If an observer makes a prediction about a system we must be able to reliably translate said prediction to other observers so we can compare results. This is rooted on the need for science to be replicable.

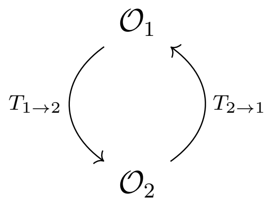
It is of fundamental interest to study what kind of transformations between observers preserve physics. For theses transformations to be consistent a couple of requirements must be satisfied

1. For any observer $\mathcal{O}_a$, the transformation unto themselves must do nothing. That is, there must exist a null transformation.



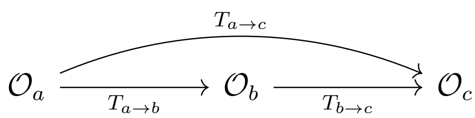
(3.0.1)

2. For any pair of observers $\mathcal{O}_1$ and $\mathcal{O}_2$ such that there is a transformation $T_{1\to 2} : \mathcal{O}_1 \to \mathcal{O}_2$ there must exist a transformation $T_{2\to 1} : \mathcal{O}_2 \to \mathcal{O}_1$ that reverses the effects of $T_{1\to 2}$. That is, any transformation must have an inverse.



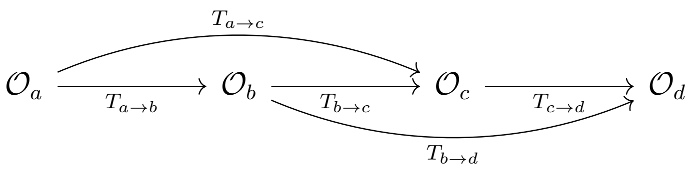
(3.0.2)

3. For any trio of observers $\mathcal{O}_1$, O_2 and $\mathcal{O}_3$, the composition of transformations $T_{a\to b} \circ T_{b\to c}$ must form the transformation $T_{a\to c}$. Where $a, b \in \{1, 2, 3\}$. This is, the composition is a closed operation.



(3.0.3)

4. For any quartet of observers $\mathcal{O}_1$, O_2, $\mathcal{O}_3$ and $\mathcal{O}_4$, it must not matter in which order the composition of transformations $T_{a\to b} \circ T_{b\to c} \circ T_{c\to d}$ is carried out. Where $a, b, c, d \in \{1, 2, 3, 4\}$. This is nothing but associativity.



(3.0.4)

Those are precisely the group axioms. The set of transformations along with the composition between them $(T, \circ)$ forms a group.

While this is true for any set of transformations between observers, to properly define the Poincaré group we need a little more structure.

3.1 Spacetime structure

A Lorentzian manifold is a pair (M_L, g), where M_L is a $(d+1)$-dimensional smooth manifold and g is a pseudo-Riemannian metric of signature $(- + + \cdots +)$. A pseudo-Riemannian metric is a non-degenerate, symmetric bilinear form $g : M_L \to T^*M_L \otimes T^*M_L$[1]. Let $x : U \subseteq M_L \to \mathbb{R}^{d+1}$ be a chart and $X, Y \in \Gamma(TM_L)$ be smooth vector fields, then at any point $p \in M_L$

$$\begin{aligned} g(p)(X,Y) = g(p)(Y,X) &= g(p)\left(X^\mu \frac{\partial}{\partial x^\mu}, Y^\nu \frac{\partial}{\partial x^\nu}\right) \\ &= X^\mu Y^\nu g(p)\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) = g(p)_{\mu\nu} X^\mu Y^\nu. \end{aligned} \tag{3.1.1}$$

For simplicity on notation, the dependence on the point p is usually omitted. In any given chart (U, x) we can write the metric g as $g(p)_{\mu\nu}\, dx^\mu \otimes dx^\nu$, where $g(p)_{\mu\nu}$ are real numbers and the matrix formed by them is symmetric and non-degenerate. Using this, it is always possible to find another chart[2] (U, y) in which the coordinate representation of g is diagonal by employment of usual linear algebra methods.

$$g(p) = -a(p)dt \otimes dt + \sum_{a=1}^{3} f_a(p) dy^a \otimes dy^a, \tag{3.1.2}$$

where both a and f_a are positively defined functions over M_L and (t, y^a) are the coordinates given by the chart (U, y). Choosing the linearly independent 1-forms e^I, with $I \in \{0, 1, 2, 3\}$, $e^0 := \sqrt{a}\, dt$ and $e^a := \sqrt{f_a} dx^a$ the expression in (3.1.2) reduces to

$$g = \eta_{IJ} e^I \otimes e^J. \tag{3.1.3}$$

[1]The symbol Γ, when anteposing a set, refers to sections over said set.
[2]We are working under the assumption of having a maximal atlas.

This construction locally equips the tangent bundle TM_L with a Minkowski structure, which will be used shortly after to talk about the Poincaré group. Before that, it will prove useful to talk a little more about other structure that arises from the objects in Lorentzian geometry.

Let $\gamma : U \subseteq \mathbb{R} \to M_L$ be a path in M_L and X_γ its tangent vector. We define the length of the path γ to be the integral[3]

$$l[\gamma] := \int_U \sqrt{\left|g(X_{\gamma(\tau)}, X_{\gamma(\tau)})\right|} d\tau, \tag{3.1.4}$$

where $\tau \in U$ is a monotonically increasing parameter. We call g a metric precisely because it gives us a way to measure lengths. It's also worthwhile to mention that $g(A, B)$ is a (pseudo)-inner product between A and B, so sometimes we will be using the notation

$$\langle A, B \rangle_g := g\left(A, B\right) \tag{3.1.5}$$

in places where emphasis in this is wanted.

This equips $\Gamma\left(TM_L\right)$ with a $C^\infty\left(M_L, \mathbb{R}\right)$-valued norm $\|\cdot\|_g$ and a way of measuring angles in the usual sense an inner product does. The norm is given by

$$\|\cdot\|_g : \Gamma\left(TM_L\right) \longrightarrow C^\infty\left(M_L, \mathbb{R}\right) \tag{3.1.6}$$

$$X \longrightarrow \|X\|_g := \sqrt{\left|\langle X, X \rangle_g\right|}, \tag{3.1.7}$$

and the angle between two vector fields is defined as

[3]If you have studied relativistic classical mechanics you might recognize this functional.

$$\theta : \Gamma\left(TM_L\right) \times \Gamma\left(TM_L\right) \longrightarrow C^{\infty}\left(M_L, \mathbb{R}\right) \tag{3.1.8}$$

$$(X, Y) \longrightarrow \theta\left(X, Y\right) := \arccos\left(\frac{\langle X, Y\rangle_g}{\|X\|_g \|Y\|_g}\right). \tag{3.1.9}$$

This definition will be needed when we talk about the conformal group, so keep it in mind.

3.1.1 Induced structure

Once a choice of metric g is made, it is possible to induce other structure from it. All in service of making as few arbitrary decisions as necessary. In particular, we can define a pseudo-inner-product for p-forms, a volume form and the Hodge star operator. All of them relevant in this work.

3.1.1.1 Co-metric

A very natural question to ask once we have a pseudo-inner product on the tangent bundle TM_L is whether we can construct from it one in the cotangent bundle T^*M_L[4]. First we need to construct a map that allows us to get a single co-vector from any given vector. This is achieved by

$$\flat : \Gamma\left(TM_L\right) \longrightarrow \Gamma\left(T^*M_L\right) \tag{3.1.10}$$

$$X \longrightarrow \flat(X) := g\left(X, \cdot\right). \tag{3.1.11}$$

Remark: this can be used to express $g\left(X, Y\right)$ as $\flat\left(X\right)\left(Y\right)$. Also, the notation $X^\flat := \flat\left(X\right)$ is somewhat used to simplify this, so $g\left(X, Y\right) = \flat\left(X\right)\left(Y\right) = X^\flat\left(Y\right)$.

With this, we can construct a co-metric $\overline{g} : M_L \to TM_L \otimes TM_L$[5] by imposing the requirement that for all smooth vector fields X and Y

[4] The answer is yes, of course. And it follows the same logic as Riesz representation theorem in quantum mechanics, where given any vector $\psi \in \mathcal{H}$ in the Hilbert space we get a unique element of its dual $\mathcal{H}^*$ by the use of the inner product $l_\psi := \langle \psi, \cdot \rangle$.

[5] If we are really strict, $\overline{g}$ is a section over $\left(T^*M_L\right)^* \otimes \left(T^*M_L\right)^*$. But the double dual of a finite vector space is isomorphic to the vector space itself and we like to keep things simple.

$$\overline{g}\left(\flat(X),\flat(Y)\right) := g\left(X,Y\right). \tag{3.1.12}$$

We can, of course, express this in coordinates. Given a choice of a chart x we have $\overline{g}(dx^\mu, dx^\nu) = g^{\mu\nu}$ and for arbitrary vector fields A and B we have

$$\overline{g}\left(\flat\left(A\right),\flat\left(B\right)\right) = g(A,B) \tag{3.1.13}$$
$$\overline{g}(g(A,\cdot),g(B,\cdot)) = g_{\alpha\beta}dx^\alpha \otimes dx^\beta(A,B) \tag{3.1.14}$$
$$\overline{g}\left(g_{\mu\nu}A^\mu dx^\nu, g_{\kappa\lambda}B^\kappa dx^\lambda\right) = g_{\alpha\beta}dx^\alpha(A)dx^\beta(B) \tag{3.1.15}$$
$$g_{\mu\nu}A^\mu g_{\kappa\lambda}B^\kappa\overline{g}(dx^\nu, dx^\lambda) = g_{\alpha\beta}A^\alpha B^\beta \tag{3.1.16}$$
$$\left(g^{\nu\lambda}g_{\kappa\lambda}\right)g_{\mu\nu}A^\mu B^\kappa = g_{\alpha\beta}A^\alpha B^\beta. \tag{3.1.17}$$

Then, the requirement of equation (3.1.12) is equivalent to imposing that the matrix $[\overline{g}^{\mu\nu}]$ constructed from the components of $\overline{g}$ is the inverse of that which is constructed from the components of g. In other words, $g^\mu{}_\nu = \delta^\mu{}_\nu$. Note that this implies that for a metric expressed as that of equation (3.1.2) we get a co-metric expressed as

$$\overline{g}(p) = -\frac{1}{a(p)}\frac{\partial}{\partial t}\otimes\frac{\partial}{\partial t} + \sum_{a=1}^{3}\frac{1}{f_a(p)}\frac{\partial}{\partial x^a}\otimes\frac{\partial}{\partial x^a}. \tag{3.1.18}$$

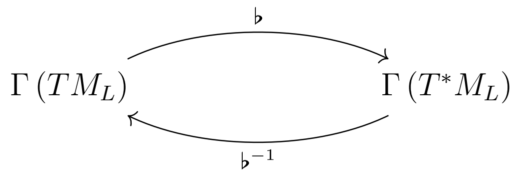
This also means we can construct a map $\flat^{-1} : \Gamma\left(T^*M_L\right) \to \Gamma\left(TM_L\right)$ by employment of the co-metric $\overline{g}$[6]

$$\flat^{-1} : \Gamma\left(T^*M_L\right) \longrightarrow \Gamma\left(TM_L\right) \tag{3.1.19}$$
$$\omega \longrightarrow \flat^{-1}(\omega) := \overline{g}\left(\omega,\cdot\right). \tag{3.1.20}$$

With this, we have a canonical isomorphism between vector fields over M_L and

[6]Hurray for the formalization of raising and lowering indices.

1-forms over M_L



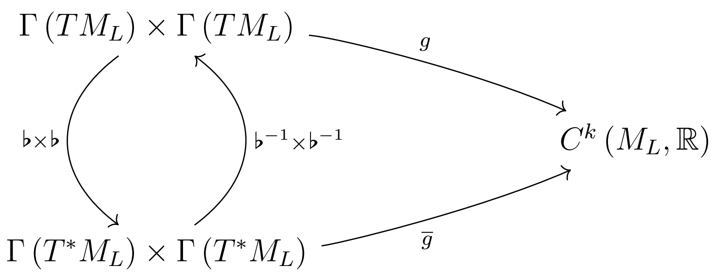
(3.1.21)

Of course, this can be generalized for tensor products of the tangent and cotangent spaces of M_L.

Finally, it may be relevant to remark that the existence of $\overline{g}$ gives us a pseudo-inner product over T^*M_L. Let α and ω be covector fields over M_L

$$\langle \alpha, \omega \rangle_{\overline{g}} := \overline{g}(\alpha, \omega). \tag{3.1.22}$$

This can be neatly summarized in the following diagram



(3.1.23)

Now, dear reader, you may be wondering why so much emphasis on this. The reason is to make it clear how much of Lorentzian geometry relies on the existence of a metric. All arrows in (3.1.23) are constructed using g. The requirement that we can go back and forth between vectors and co-vectors using $\flat$ and $\flat^{-1}$ is not met when we are dealing with degenerate metrics, which appear both in Galilean and Carrollian geometries.

3.1.1.2 Volume form

A volume form is a non-degenerate, nowhere vanishing top form over an oriented manifold[7]. Both requirements stem from wanting an object that gives a well-defined notion of volume on a manifold. Most importantly, given a pseudo-Riemannian metric g over M_L, there exists a volume form $\omega_g \in \Omega^{d+1}(M_L)$ that, in coordinate induced basis is expressed as

$$\omega_g := \sqrt{|\det g_x|}\, dx^1 \wedge \cdots \wedge dx^{d+1}, \tag{3.1.24}$$

where $\det g_x$ is the determinant of the matrix of components $[g_{\mu\nu}]$ with respect to the chart (U, x).

Now, $\det g$ has a very strong dependence on coordinates and whenever one defines something in such a way, care must be put into verifying that the choice of coordinates does not matter. Given another chart $y : V \subseteq M_L \to \mathbb{R}^{d+1}$ with non-empty intersection $U \cap V$ with $x : U \subseteq M_L \to \mathbb{R}^{d+1}$ it should be the case we can transform from (3.1.24) to

$$\omega_g = \sqrt{|\det g_y|}\, dy^1 \wedge \cdots \wedge dy^{d+1} \tag{3.1.25}$$

in the intersection $U \cap V$.

Let $\phi : x(U \cap V) \to y(U \cap V)$ be the chart transition map $y \circ x^{-1}$ and $\Phi = (\phi^*)^{-1}$. Then any 1-form $\alpha \in T^*(x(U \cap V))$ transforms under ϕ as $\Phi\alpha$. This can be seen in the following diagram

[7]An orientation is needed to have a properly defined notion of oriented areas and volumes, which are used in integration theory in general and in Stokes theorem in particular.

$$\begin{array}{ccccc}
 & & \overset{\phi^*}{\frown} & & \\
T^*\left(y\left(U\cap V\right)\right)\subseteq\mathbb{R}^{2n} & \xleftarrow{T^*y} & T^*\left(U\cap V\right) & \xrightarrow{T^*x} & T^*\left(x\left(U\cap V\right)\right)\subseteq\mathbb{R}^{2n} \\
\Big\downarrow{\scriptstyle\pi_y} & & \Big\downarrow{\scriptstyle\pi} & & \Big\downarrow{\scriptstyle\pi_x} \\
y\left(U\cap V\right)\subseteq\mathbb{R}^{n} & \xleftarrow[y]{} & \left(U\cap V\right)\subseteq M_L & \xrightarrow[x]{} & x\left(U\cap V\right)\subseteq\mathbb{R}^{n} \\
 & & \underset{\phi}{\smile} & &
\end{array} \tag{3.1.26}$$

where $n=d+1$. In the particular case of 1-forms dx^μ we have

$$\phi^* dx^\mu = \frac{\partial x^\mu}{\partial y^\nu} dy^\nu, \tag{3.1.27}$$

therefore, using the distributive property of the pullback we get that for a top form

$$\begin{aligned}
\Phi\left(dx^1\wedge\cdots\wedge dx^{d+1}\right) &= \Phi\, dx^1\wedge\cdots\wedge\Phi\, dx^{d+1} && (3.1.28)\\
&= \left(\frac{\partial x^1}{\partial y^\mu}dy^\mu\right)\wedge\cdots\wedge\frac{\partial x^{d+1}}{\partial y^\nu}dy^\nu && (3.1.29)\\
&= \det\left(\frac{\partial x}{\partial y}\right) dy^1\wedge\cdots\wedge dy^{d+1} && (3.1.30)\\
&= \det\left(\Phi\right) dy^1\wedge\cdots\wedge dy^{d+1}, && (3.1.31)
\end{aligned}$$

and

$$\begin{aligned}
\sqrt{\left|\det g_{x\,\mu\nu}\right|} &= \sqrt{\left|\det\left(g_{y\,\alpha\beta}\frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu}\right)\right|} && (3.1.32)\\
&= \sqrt{\left|\det g_y \det\left(\Phi^{-1}\right)^2\right|} && (3.1.33)\\
&= \sqrt{\left|\det g_y\right|}\left|\det\left(\Phi^{-1}\right)\right|. && (3.1.34)
\end{aligned}$$

Putting it all together and using $\det(A)\det(A^{-1}) = 1$ we get

$$\sqrt{|\det g_x|}\, dx^1 \wedge \cdots \wedge dx^{d+1} = \operatorname{sign}(\det \Phi) \sqrt{|\det g_y|}\, dy^1 \wedge \cdots \wedge dy^{d+1}. \quad (3.1.35)$$

But since we are working in an orientable manifold, all chart transition functions must preserve the orientation. Therefore $\operatorname{sign}(\det \Phi) = 1$. And so our wonky definition of a volume form ω_g is, in fact, chart independent and we can simplify the notation as $\omega_g = \sqrt{|\det g|} dx^1 \wedge \cdots \wedge dx^{d+1}$. Notice that we can also write it as $\omega_g = e^1 \wedge \cdots \wedge e^{d+1}$.

This is quite important for it allows us to define integration over functions $f \in C^\infty(M_L, \mathbb{R})$. Let (U, x) be a chart, we define the integration of f over U as

$$\int_U f := \int_U f\omega_g, \quad (3.1.36)$$

where the second term is the usual integration of a top form over a sub-manifold[8]. To extend this notion of integration over the entire manifold, a partition of unity is needed. The reason behind wanting an integration theory of scalar functions over a Lorentzian manifold is to be able to have action principles in terms of Lagrangians.

3.1.1.3 Pseudo-inner-product for p-forms

So far we've constructed a $C^\infty(M_L)$-valued pseudo-inner product on the cotangent bundle T^*M_L from the one in the tangent bundle TM_L. A natural question is whether it is possible to extend this to p-forms $\alpha \in \Omega^p(M_L)$. The answer to this question is, of course, yes.

Consider the function

[8]i.e. $\int_U f\omega_g = \int_{x(U)} f(x)\sqrt{|\det g|} d^{d+1}x$

$$\langle \cdot, \cdot \rangle : \Omega^p (M_L) \times \Omega^p (M_L) \to C^\infty (M_L)$$
$$(a, b) \to \langle a, b \rangle := a_{\alpha_1 \dots \alpha_p} \bar{g}^{\alpha_1 \beta_1} \dots \bar{g}^{\alpha_p \beta_p} b_{\beta_1 \dots \beta_p}. \quad (3.1.37)$$

For this to be a valid notion of a pseudo-inner product we need to check that it's invariant under change of coordinates, it's bilinear and non-degenerate. Coordinate invariance is easily checked. So is bilinearity since

$$\langle a, b + c \rangle = a_{\alpha_1 \dots \alpha_p} \bar{g}^{\alpha_1 \beta_1} \dots \bar{g}^{\alpha_p \beta_p} \left(b_{\beta_1 \dots \beta_p} + c_{\beta_1 \dots \beta_p} \right) \quad (3.1.38)$$
$$= a_{\alpha_1 \dots \alpha_p} \bar{g}^{\alpha_1 \beta_1} \dots \bar{g}^{\alpha_p \beta_p} b_{\beta_1 \dots \beta_p} + a_{\alpha_1 \dots \alpha_p} \bar{g}^{\alpha_1 \beta_1} \dots \bar{g}^{\alpha_p \beta_p} c_{\beta_1 \dots \beta_p} \quad (3.1.39)$$
$$= \langle a, b \rangle + \langle a, c \rangle. \quad (3.1.40)$$

And the bilinear form $\langle \cdot, \cdot \rangle$ is symmetric

$$\langle a, b \rangle = a_{\alpha_1 \dots \alpha_p} \bar{g}^{\alpha_1 \beta_1} \dots \bar{g}^{\alpha_p \beta_p} b_{\beta_1 \dots \beta_p} = b_{\beta_1 \dots \beta_p} \bar{g}^{\alpha_1 \beta_1} \dots \bar{g}^{\alpha_p \beta_p} a_{\alpha_1 \dots \alpha_p} = \langle b, a \rangle. \quad (3.1.41)$$

Bilinearity checked it only rest to check non-degeneracy, which is a direct consequence of non-degeneracy of the metric g.

3.1.1.4 Hodge dual star operator

The main topic of this work is electrodynamics, which is a $U(1)$ gauge theory. In gauge theories we have three main objects to work with: the connection A, its curvature F and a current J. While it is true it is possible to construct an action in four dimensions with only this for a $U(1)$ theory, namely

$$\int_U F \wedge A \wedge J, \quad (3.1.42)$$

to build electrodynamics as we know it we need a way to have a non-vanishing quadratic term of F and this is not possible without introducing a metric structure since $F \wedge F = d(A \wedge F)$ is a border term and therefore yields no dynamics in the bulk. Here is where the Hodge dual star operator comes into play.

The Hodge dual star operator is the unique map[9] $\star : \Omega^k(M_L) \to \Omega^{d+1-k}(M_L)$ such that for any $\alpha, \beta \in \Omega^p(M_L)$

$$\alpha \wedge \star\beta = -\langle \alpha, \beta \rangle \omega_g. \tag{3.1.43}$$

In coordinates, the Hodge dual of a p-form $\alpha = \frac{1}{p!}\alpha_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$ in a $(d+1)$-dimensional Lorentzian manifold is

$$(\star\alpha)_{\nu_{p+1} \dots \nu_{d+1}} = \frac{1}{p!(d+1-p)!}\sqrt{|\det g|}\, \alpha_{\mu_1 \dots \mu_p} \bar{g}^{\mu_1 \nu_1} \dots \bar{g}^{\mu_p \nu_p} \epsilon_{\nu_1 \dots \nu_p \nu_{p+1} \dots \nu_{d+1}}, \tag{3.1.44}$$

where $\epsilon_{\mu_1 \dots \mu_{d+1}}$ is the completely antisymmetric Levi-Civita symbol with orientation $\epsilon_{012\dots d} = 1$.

3.1.2 Causal structure

Causal structure refers to a (local) classification of non-zero vectors $X \in T_p M_L$ in the tangent space $T_p M_L$ into space-like, time-like or null if[10] $g(X, X) > 0$, $g(X, X) < 0$ or $g(X, X) = 0$, respectively. This is usually represented as light-cones, with the interior of said cone corresponding to the sector $U^+ \in M_L$ which can be reached by light signals emitted at the point $p \in M_L$ or the sector $U^- \in M_L$ that can reach the point p with light signals. A smooth curve $\gamma : U \subseteq \mathbb{R} \to \gamma(U) \subset M_L$ is called timelike, spacelike or null if its tangent vector $\dot{\gamma} \in T\gamma(U)$ is timelike, spacelike or null, respectively for every point along the curve γ. A vector field can also be called timelike, spacelike or null if at every point it satisfies the appropriate criterion.

[9] The case of Riemannian geometry has $\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \omega_g$.

[10] Causal structures can be defined in a purely topological manner without referring to a metric, as was shown in Rainer (1999).

Furthermore, a Lorentzian manifold is said to be time-oriented if it admits a continuous, nowhere vanishing, timelike vector field τ. This vector field is used to classify timelike vector fields as future-directed or past-directed.

3.1.3 Minkowski spacetime

Let (M, η) be a Lorentzian manifold. We say that (M, η) is Minkowski spacetime if and only if

1. M is the Cartesian space $\mathbb{R}^{d+1}$.
2. η is, in standard coordinates, characterized by $\mathrm{diag}(-1, 1, \ldots, 1)$.

A very important and immediate consequence of this definition is that Minkowski spacetime is flat. i.e. its curvature vanishes.

3.2 Group structure and decomposition

The isometries of a spacetime structure (M, g) are isomorphisms $a : M \to M$ that preserve this structure. For this to happen it is needed that the metric remains invariant under the pullback a^* of a

$$a^* g = g. \tag{3.2.1}$$

In other words, for all vector fields $X, Y \in TM$ $(a^*g)(X, Y) = g(a_* X, a_* Y) = g(X, Y)$. Consider the case where the map a is the flow $h_\lambda^X : M \to M$ of a vector field X, then we can construct an equivalent criterion via a differential quotient

$$\left(h_\lambda^X\right)^* g - g = 0 \tag{3.2.2}$$

$$\lim_{\lambda \to 0} \frac{\left(h_\lambda^X\right)^* g - g}{\lambda} = 0 \tag{3.2.3}$$

$$\mathcal{L}_X g = 0, \tag{3.2.4}$$

where $\mathcal{L}_X g$ is the Lie derivative of g with respect to the vector field X. This makes easy to check whether a vector field X generates a symmetry transformation of the metric g. The real vector space formed by all vector fields satisfying (3.2.4) constitute a Lie sub-algebra of the Lie algebra of vector fields over M Sontz (2015) with Lie bracket

$$[\cdot,\cdot] : \Gamma(TM) \times \Gamma(M) \to \Gamma(TM) \tag{3.2.5}$$

$$(A,B) \to [A,B] := AB - BA. \tag{3.2.6}$$

Let A and B be vector fields that generate isometries of g and α a real number, then their linear combination $\alpha A + B$ is an isometry of g

$$\mathcal{L}_{\alpha A+B}\, g = \alpha \mathcal{L}_A g + \mathcal{L}_B g = 0, \tag{3.2.7}$$

this is also the case of their commutator $[A,B]$

$$\mathcal{L}_{[A,B]}\, g = \mathcal{L}_A \mathcal{L}_B g - \mathcal{L}_B \mathcal{L}_A g = 0. \tag{3.2.8}$$

Just as these form an algebra, the isometries themselves constitute a group with product given by the function composition. Not all isometries can be obtained through exponentiation $a = \exp(tX)$ though, only those connected to the identity.

The Poincaré group $\mathrm{ISO}(d,1)$ is the group of isometries of $(d+1)$-dimensional Minkowski spacetime. In this work we are mostly concerned with 4-dimensional geometry[11], therefore we will be dealing with the group $\mathrm{ISO}(3,1)$ and its algebra $\mathfrak{iso}(3,1)$. Said algebra consists of vector fields $X \in \Gamma(TM_L)$ satisfying (3.2.4), this is

$$\mathfrak{iso}(3,1) = \{ X \in \Gamma(TM_L) |\, \mathcal{L}_X \eta = 0 \}. \tag{3.2.9}$$

[11]Because our classical world is 4-dimensional.

3.2.1 Spatial translations

Each spatial translation is a 1-parameter subgroup of the Poincaré group with generator[12]

$$P_A := \frac{\partial}{\partial x^A}. \tag{3.2.10}$$

For checking whether it belongs to the algebra $\mathfrak{iso}(3,1)$, it is convenient to use the following properties of the Lie derivative

$$\mathcal{L}_X (S \otimes T) = (\mathcal{L}_X S) \otimes T + S \otimes (\mathcal{L}_X T) \qquad \mathcal{L}_X \omega = d i_X \omega + i_X d\omega, \tag{3.2.11}$$

where S and T are tensor fields over M_L and ω is a p-form. Using these two properties, the calculation of the necessary Lie derivatives is quite straightforward

$$\begin{aligned}
\mathcal{L}_{P_A} \eta &= \mathcal{L}_{P_A} (\eta_{\mu\nu} dx^\mu \otimes dx^\nu) && (3.2.12)\\
&= \eta_{\mu\nu} (\mathcal{L}_{P_A} dx^\mu) \otimes dx^\nu + \eta_{\mu\nu} dx^\mu \otimes (\mathcal{L}_{P_A} dx^\nu) && (3.2.13)\\
&= \eta_{\mu\nu} d (dx^\mu (P_A)) \otimes dx^\nu + \eta_{\mu\nu} dx^\mu \otimes d (dx^\nu (P_A)) && (3.2.14)\\
&= 0. && (3.2.15)
\end{aligned}$$

So $P_A \in \mathfrak{iso}(3,1)$, we now need to reconstruct its isometry. This is done by the usual method of finding the integral curves of vector fields P_A and then using them to construct the flows that serve as the action of the 1-parameter subgroup with generator P_A.

[12] From this point onward, uppercase indices will indicate space, running from 1 to 3 and greek ones will indicate space-time, running from zero to three.

Let X be a vector field in the algebra, the flow h^X of the vector field X is a function

$$h^X : \mathbb{R} \times M \to M \tag{3.2.16}$$

$$(\lambda, m) \to h^X(\lambda, m) := \gamma_m(\lambda), \tag{3.2.17}$$

where γ_m is a solution to the following system of ordinary differential equations

$$\dot{\gamma}_m(\lambda) = X_{\gamma_m(\lambda)} \tag{3.2.18}$$

with initial conditions $\gamma_m(0) = m$ and where $\dot{\gamma}_m$ is the tangent vector to the curve γ and X_{γ_m} is the vector field X evaluated along said curve.

For the case of P_1, this equation reads as

$$\dot{x}(\lambda)\frac{\partial}{\partial x} + \dot{y}(\lambda)\frac{\partial}{\partial y} + \dot{z}(\lambda)\frac{\partial}{\partial z} + \dot{t}(\lambda)\frac{\partial}{\partial t} = \frac{\partial}{\partial x}. \tag{3.2.19}$$

Using linear independence and solving the equations with initial conditions $x^\mu(0) = x_0^\mu$, we conclude that

$$h^{P_1}(\lambda, t, x, y, z) = (t, x + \lambda, y, z). \tag{3.2.20}$$

It follows this is also the case for both P_2 and P_3, with

$$h^{P_2}(\lambda, t, x, y, z) = (t, x, y + \lambda, z) \quad h^{P_3}(\lambda, t, x, y, z) = (t, x, y, z + \lambda). \tag{3.2.21}$$

The following picture are shows the flow lines of h^{P_1}, with the horizontal axis being x and the vertical axis being the time t[13].

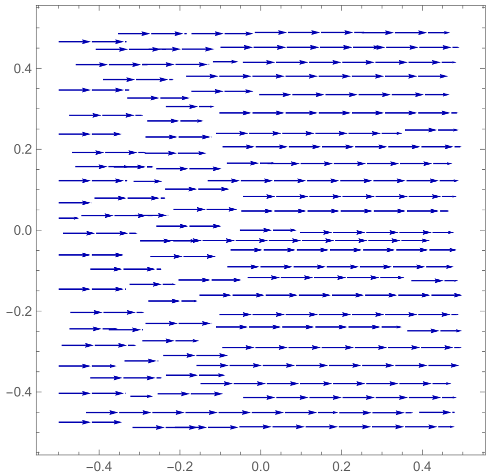


Figure 3.2.1: Integral curves of P_1.

3.2.2 Time translations

Time translations are also a 1-parameter subgroup of the Poincaré group and it shares the exact same shape as spatial translations. Which makes sense since, in the context of special relativity, time and space are just labels to one sole thing: space-time.

The generator $H \in \mathfrak{iso}(3,1)$ of time translations is

$$H := \frac{1}{c}\frac{\partial}{\partial t}. \tag{3.2.22}$$

We proceed to check whether it generates an isometry via Lie derivative of the Minkowski metric

$$\begin{aligned}
\mathcal{L}_H \eta &= \mathcal{L}_H \left(\eta_{\mu\nu} dx^\mu \otimes dx^\nu\right) && (3.2.23)\\
&= \eta_{\mu\nu} \left(\mathcal{L}_H dx^\mu\right) \otimes dx^\nu + \eta_{\mu\nu} dx^\mu \otimes \left(\mathcal{L}_H dx^\nu\right) && (3.2.24)\\
&= \eta_{\mu\nu} d\left(dx^\mu (H)\right) \otimes dx^\nu + \eta_{\mu\nu} dx^\mu \otimes d\left(dx^\nu (H)\right) && (3.2.25)\\
&= 0. && (3.2.26)
\end{aligned}$$

[13]This is the usual way of representing space-time in special relativity.

We now consider the flow equation for time translations

$$\dot{\gamma}_m(\lambda) = H_{\gamma_m(\lambda)} \tag{3.2.27}$$

$$\dot{x}(\lambda)\frac{\partial}{\partial x} + \dot{y}(\lambda)\frac{\partial}{\partial y} + \dot{z}(\lambda)\frac{\partial}{\partial z} + \dot{t}(\lambda)\frac{\partial}{\partial t} = \frac{1}{c}\frac{\partial}{\partial t}. \tag{3.2.28}$$

The solution of this differential problem with initial conditions $x^\mu(0) = x_0^\mu$ is the curve $\gamma_{(t_0,x_0,y_0,z_0)}(\lambda) = (t_0 + \lambda/c, x_0, y_0, z_0)$. Therefore the flow of H is

$$h^H(\lambda, t, x, y, z) = (t + \lambda/c, x, y, z). \tag{3.2.29}$$

The isometries h^{P_A} and h^H are each a representation of the additive group $(\mathbb{R}, +)$. The integral lines of H are represented below, with time as the vertical axis and space as the horizontal axis.

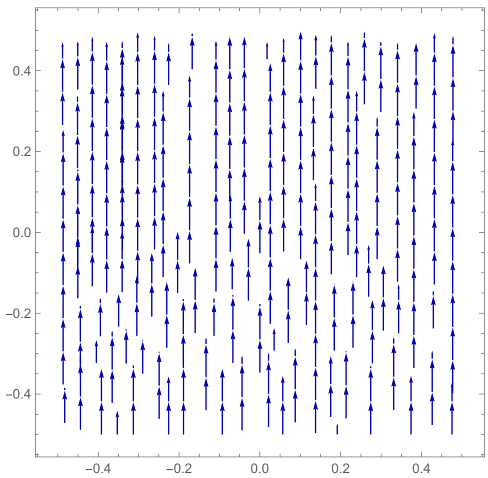


Figure 3.2.2: Integral curves of H.

3.2.3 Spatial rotations

Each spatial rotation is a 1-parameter subgroup of the Poincaré group with generator $J_A \in \mathfrak{iso}(3, 1)$ given by

$$J_A := \epsilon_{ABC} x^B \frac{\partial}{\partial x^C}, \tag{3.2.30}$$

where ϵ_{ABC} is the 3-dimensional Levi-Civita symbol with $\epsilon_{123} = 1$.

We check that it satisfies the condition (3.2.4), as we have already done with spatial and time translations

$$\begin{aligned}
\mathcal{L}_{J_A}\eta &= \mathcal{L}_{J_A}\left(\eta_{\mu\nu} dx^\mu \otimes dx^\nu\right) &(3.2.31)\\
&= \eta_{\mu\nu}\left(\mathcal{L}_{J_A} dx^\mu\right) \otimes dx^\nu + \eta_{\mu\nu} x d^\mu \otimes \left(\mathcal{L}_{J_A} dx^\nu\right) &(3.2.32)\\
&= \eta_{\mu\nu} d\left(dx^\mu\left(J_A\right)\right) \otimes dx^\nu + \eta_{\mu\nu} dx^\mu \otimes d\left(dx^\nu\left(J_A\right)\right) &(3.2.33)\\
&= \eta_{C\nu}\, \epsilon_{ABC} dx^B \otimes dx^\nu + \eta_{\mu C}\, \epsilon_{ABC} dx^\mu \otimes dx^B &(3.2.34)\\
&= \eta_{CD}\, \epsilon_{ABC} dx^B \otimes dx^D + \eta_{DC}\, \epsilon_{ABC} dx^D \otimes dx^B &(3.2.35)\\
&= 0, &(3.2.36)
\end{aligned}$$

where the last line corresponds to the symmetrization of an antisymmetric object and is therefore zero. We construct the flows h^{J_A} by solving the differential equations

$$\dot{\gamma}_m(\lambda) = J_{\gamma_m(\lambda)}. \tag{3.2.37}$$

We start with J_1, whose differential equation corresponds to

$$\dot{x}(\lambda)\frac{\partial}{\partial x} + \dot{y}(\lambda)\frac{\partial}{\partial y} + \dot{z}(\lambda)\frac{\partial}{\partial z} + \dot{t}(\lambda)\frac{\partial}{\partial t} = z(\lambda)\frac{\partial}{\partial y} - y(\lambda)\frac{\partial}{\partial z}. \tag{3.2.38}$$

Solving this equation we get the flow

$$h^{J_1}\left(\lambda, t, x, y, z\right) = \left(t, x, y\cos\lambda + z\sin\lambda, z\cos\lambda - y\sin\lambda\right), \tag{3.2.39}$$

which corresponds to a rotation of angle $\lambda \in \mathbb{R}$ with respect to the x-axis, which can be thought of as the action of the additive group $(S^1, +)$. We repeat the procedure for J_2

$$\dot{x}(\lambda)\frac{\partial}{\partial x} + \dot{y}(\lambda)\frac{\partial}{\partial y} + \dot{z}(\lambda)\frac{\partial}{\partial z} + \dot{t}(\lambda)\frac{\partial}{\partial t} = x(\lambda)\frac{\partial}{\partial z} - z(\lambda)\frac{\partial}{\partial x}. \tag{3.2.40}$$

Solving this equation we construct J_2's flow

$$h^{J_2}(\lambda, t, x, y, z) = (t, x\cos\lambda - z\sin\lambda, y, z\cos\lambda + x\sin\lambda), \tag{3.2.41}$$

which corresponds to a rotation of angle $\lambda \in \mathbb{R}$ with respect to the y-axis, which can be thought of as the action of the additive group $(S^1, +)$. Finally, the differential equations for J_3 are as follows

$$\dot{x}(\lambda)\frac{\partial}{\partial x} + \dot{y}(\lambda)\frac{\partial}{\partial y} + \dot{z}(\lambda)\frac{\partial}{\partial z} + \dot{t}(\lambda)\frac{\partial}{\partial t} = y(\lambda)\frac{\partial}{\partial x} - x(\lambda)\frac{\partial}{\partial y}. \tag{3.2.42}$$

We use the solution of this equation to construct the flow

$$h^{J_3}(\lambda, t, x, y, z) = (t, x\cos\lambda + y\sin\lambda, y\cos\lambda - x\sin\lambda, z), \tag{3.2.43}$$

which corresponds to a rotation of angle $\lambda \in \mathbb{R}$ with respect to the z-axis, which can be thought of as the action of the additive group $(S^1, +)$. The three rotations put together are an action of the rotation group SO(3)[14].

3.2.4 Boosts

Each boost is a one parameter subgroup of the Poincaré group that mixes space and time in pretty much the same fashion as spatial rotation mixes space. Boosts have generators $B_A \in \mathfrak{iso}(3,1)$ given by

[14]Recall this only allows us to recover the connected part to the identity.

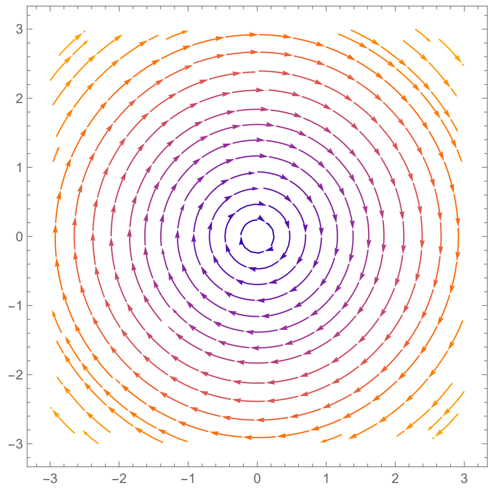


Figure 3.2.3: Integral curves of J_1.

$$B_A := ct\frac{\partial}{\partial x^A} + \frac{x_A}{c}\frac{\partial}{\partial t}. \tag{3.2.44}$$

It is noteworthy to mention x_A is simply a shorthand to $\eta_{AB}x^B$ and since we cleverly chose the signature this is the same as x^A and should *not* be confused with the musical isomorphisms since this is not a vector nor a covector. We verify $B_A \in \mathfrak{iso}\,(3,1)$ as follows

$$\begin{aligned}
\mathcal{L}_{B_A}\eta &= \mathcal{L}_{B_A}\left(\eta_{\mu\nu}dx^\mu \otimes dx^\nu\right) && (3.2.45)\\
&= \eta_{\mu\nu}\left(\mathcal{L}_{B_A}dx^\mu\right) \otimes dx^\nu + \eta_{\mu\nu}dx^\mu \otimes \left(\mathcal{L}_{B_A}dx^\nu\right) && (3.2.46)\\
&= \eta_{\mu\nu}d\left(dx^\mu\left(B_A\right)\right) \otimes dx^\nu + \eta_{\mu\nu}dx^\mu \otimes d\left(dx^\nu\left(B_A\right)\right) && (3.2.47)\\
&= \eta_{A\nu}cdt \otimes dx^\nu + \frac{1}{c}\eta_{0\nu}dx^A \otimes dx^\nu + \eta_{\mu A}dx^\mu \otimes cdt + \frac{1}{c}\eta_{\mu 0}dx^\mu \otimes dx^A && (3.2.48)\\
&= \eta_{AB}cdt \otimes dx^B - cdx_A \otimes dt + \eta_{AB}cdx^B \otimes dt - cdt \otimes dx_A && (3.2.49)\\
&= 0. && (3.2.50)
\end{aligned}$$

Next we construct the flows by solving the appropriate differential equations. For B_1 this equation is

$$\dot{x}(\lambda)\frac{\partial}{\partial x} + \dot{y}(\lambda)\frac{\partial}{\partial y} + \dot{z}(\lambda)\frac{\partial}{\partial z} + \dot{t}(\lambda)\frac{\partial}{\partial t} = ct\frac{\partial}{\partial x} + \frac{x}{c}\frac{\partial}{\partial t}. \tag{3.2.51}$$

Solving this we conclude that the flow of B_1 is

$$h^{B_1}(\lambda, t, x, y, z) = \left(t\cosh(c\lambda) + x\sinh(c\lambda)/c, x\cosh(c\lambda) + ct\sinh(c\lambda), y, z\right). \tag{3.2.52}$$

For the vector B_2 we've got the following equation

$$\dot{x}(\lambda)\frac{\partial}{\partial x} + \dot{y}(\lambda)\frac{\partial}{\partial y} + \dot{z}(\lambda)\frac{\partial}{\partial z} + \dot{t}(\lambda)\frac{\partial}{\partial t} = ct\frac{\partial}{\partial y} + \frac{y}{c}\frac{\partial}{\partial t}, \tag{3.2.53}$$

so the flow of B_2 is

$$h^{B_2}(\lambda, t, x, y, z) = \left(t\cosh(c\lambda) + y\sinh(c\lambda)/c, x, y\cosh(c\lambda) + ct\sinh(c\lambda), z\right). \tag{3.2.54}$$

Finally, the remaining equation to solve is

$$\dot{x}(\lambda)\frac{\partial}{\partial x} + \dot{y}(\lambda)\frac{\partial}{\partial y} + \dot{z}(\lambda)\frac{\partial}{\partial z} + \dot{t}(\lambda)\frac{\partial}{\partial t} = ct\frac{\partial}{\partial z} + \frac{z}{c}\frac{\partial}{\partial t}, \tag{3.2.55}$$

and the flow of B_3 is

$$h^{B_3}(\lambda, t, x, y, z) = \left(t\cosh(c\lambda) + z\sinh(c\lambda)/c, x, y, z\cosh(c\lambda) + ct\sinh(c\lambda)\right). \tag{3.2.56}$$

We illustrate how these three flows work in the following graph, which corresponds to the flow lines of B_1

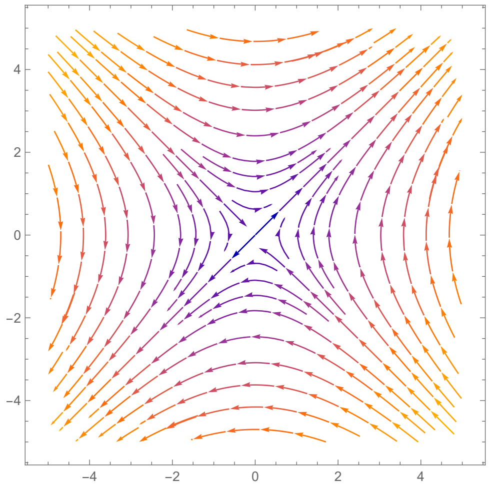


Figure 3.2.4: Integral curves of B_1.

The three spatial rotations h^{J_A} together with the three boosts h^{B_A} correspond to the action of the group $SO\,(3,1)$.

3.3 Algebra

Having the vector fields that generate $\mathfrak{iso}\,(3,1)$, it is possible to construct their algebra by taking the differential geometric commutator between them. Recall that the algebra $\mathfrak{iso}\,(3,1)$ is generated through the linear combination of P_A, H, J_A and B_A, this is

$$\mathfrak{iso}\,(3,1) = \text{span}_{\mathbb{R}}\mathcal{A}, \tag{3.3.1}$$

where $\mathcal{A} := \{P_A, H, J_A, B_A\}_{A=\{1,2,3\}}$. This, together with the following commutators

$$[P_A, P_B] = 0 \quad [P_A, H] = 0 \quad [P_A, J_B] = \epsilon_{ABC} J_C \quad [P_A, B_B] = \delta_{AB} H \tag{3.3.2}$$

$$[H, J_A] = 0 \quad [H, B_A] = cP_A \quad [J_A, J_B] = \epsilon_{ABC} J_C \quad [J_A, B_B] = \epsilon_{ABC} B_C, \tag{3.3.3}$$

constitutes the Poincaré algebra. Generators $X \in \mathfrak{iso}(3,1)$ constitute all isometries of Minkowski metric.

3.4 Conformal extension

The Poincaré group are all transformations that preserve the distance defined by the Minkowski metric. This can be extended to consider all symmetry transformations that preserve the angles between vector fields as defined in (3.1.8) associated to η. In other words, transformations such that the angle

$$\theta : \Gamma(TM) \times \Gamma(TM) \longrightarrow C^{\infty}(M, \mathbb{R}) \tag{3.4.1}$$

$$(X, Y) \longrightarrow \theta(X, Y) := \arccos\left(\frac{\langle X, Y\rangle_{\eta}}{\|X\|_{\eta}\|Y\|_{\eta}}\right) \tag{3.4.2}$$

remains invariant. Let the map $a : M \longrightarrow M$ be a map such that $a^{*}\eta = \Omega^{2}\eta$, where $\Omega : M \longrightarrow \mathbb{R}$ is a real-valued non-zero function, then for all vector fields $X, Y \in \Gamma(TM)$ we have

$$(a^{*}\theta)(X, Y) = \theta(a_{*}X, a_{*}Y) \tag{3.4.3}$$

$$= \arccos\left(\frac{\langle a_{*}X, a_{*}Y\rangle_{\eta}}{\|a_{*}X\|_{\eta}\|a_{*}Y\|_{\eta}}\right) \tag{3.4.4}$$

$$= \arccos\left(\frac{\Omega^{2}\langle X, Y\rangle_{\eta}}{\Omega^{2}\|X\|_{\eta}\|Y\|_{\eta}}\right) \tag{3.4.5}$$

$$= \theta(X, Y). \tag{3.4.6}$$

It follows these transformations are part of the conformal extension. Furthermore, if $a^{*}\eta = \Omega^{2}\eta$ it follows $a_{*}\bar{\eta} = \Omega^{-2}\bar{\eta}$[15], which means $a^{*}(\eta \otimes \bar{\eta}) = (a^{*}\eta) \otimes (a_{*}\bar{\eta}) = \eta \otimes \bar{\eta}$. Which can be used to construct a useful criterion to identifying the generators of conformal symmetries via differential quotient. Let $a = \exp(\lambda X)$ for a real parameter λ and a vector field X, then

[15] Recall $\bar{\eta}$ is the co-metric of η as defined in (3.1.18). This object is usually referred to as inverse metric but this name is preferably to be avoided.

$$\exp\left(\lambda X\right)^{*}\left(\eta\otimes\bar{\eta}\right)-\eta\otimes\bar{\eta}=0 \tag{3.4.7}$$

$$\lim_{\lambda\to 0}\frac{\exp\left(\lambda X\right)^{*}\left(\eta\otimes\bar{\eta}\right)-\eta\otimes\bar{\eta}}{\lambda}=0 \tag{3.4.8}$$

$$L_{X}\left(\eta\otimes\bar{\eta}\right)=0. \tag{3.4.9}$$

All vector fields in $X\in\mathfrak{iso}(3,1)$ satisfy this criterion since $L_X\eta=0$ and $L_X\bar{\eta}=0$. There are two additional generators that extend this algebra, space-time dilations D and special conformal transformations S_μ. Dilations have generator

$$D=x^{A}\frac{\partial}{\partial x^{A}}+t\frac{\partial}{\partial t}. \tag{3.4.10}$$

The Lie derivative of the metric η with respect to the dilation generator D is given by

$$\begin{aligned}
L_D\eta =& L_D\left(\eta_{\mu\nu}dx^{\mu}\otimes dx^{\nu}\right) && (3.4.11)\\
=& L_D\left(-c^2dt\otimes dt+\delta_{AB}dx^{A}\otimes dx^{B}\right) && (3.4.12)\\
=& -c^2d\left(dt\left(D\right)\right)\otimes dt-c^2dt\otimes d\left(dt\left(D\right)\right)+\delta_{AB}d\left(dx^{A}\left(D\right)\right)\otimes dx^{B}\\
&+\delta_{AB}dx^{A}\otimes d\left(dx^{B}\left(D\right)\right) && (3.4.13)\\
=& 2\left(-c^2dt\otimes dt+\delta_{AB}dx^{A}\otimes dx^{B}\right) && (3.4.14)\\
=& 2\eta. && (3.4.15)
\end{aligned}$$

The Lie derivative of $\tilde{\eta}$ with respect to D can be proven to be $L_D\tilde{\eta}=-2\,\tilde{\eta}$, therefore $L_D\left(\eta\otimes\tilde{\eta}\right)=0$ and condition (3.4.9) is satisfied.

The symmetry transformation associated with this vector field is obtained after solving the system of ordinary differential equations

$$\dot{x}(\lambda)\frac{\partial}{\partial x} + \dot{y}(\lambda)\frac{\partial}{\partial y} + \dot{z}(\lambda)\frac{\partial}{\partial z} + \dot{t}(\lambda)\frac{\partial}{\partial t} = x(\lambda)\frac{\partial}{\partial x} + y(\lambda)\frac{\partial}{\partial y} + z(\lambda)\frac{\partial}{\partial z} + t(\lambda)\frac{\partial}{\partial t}. \tag{3.4.16}$$

Solutions with initial conditions of this system of ODEs are used to build the flows that serve as the action of space-time dilations

$$h^D\left(\lambda, t, x, y, z\right) = \left(e^{\lambda}t, e^{\lambda}x, e^{\lambda}y, e^{\lambda}z\right), \tag{3.4.17}$$

which, of course, corresponds to acting with a multiplicative factor on all space-time coordinates.

Special conformal transformations are the less obvious conformal transformations. The temporal special conformal transformation is generated by the vector field S_0

$$S_0 = -2c^2tx\frac{\partial}{\partial x} - 2c^2ty\frac{\partial}{\partial y} - 2c^2tz\frac{\partial}{\partial z} - \left(c^2t^2 + x^2 + y^2 + z^2\right)\frac{\partial}{\partial t}. \tag{3.4.18}$$

The Lie derivative of the metric η with respect to the vector field S_0 is computed as follows

$$
\begin{aligned}
L_{S_0}\eta =& L_{S_0}\left(\eta_{\mu\nu}dx^{\mu}\otimes dx^{\nu}\right) && (3.4.19)\\
=& L_{S_0}\left(-c^2dt\otimes dt+\delta_{AB}dx^A\otimes dx^B\right) && (3.4.20)\\
=& -c^2d\left(dt\left(S_0\right)\right)\otimes dt-c^2dt\otimes d\left(dt\left(S_0\right)\right)+\delta_{AB}d\left(dx^A\left(S_0\right)\right)\otimes dx^B\\
&+\delta_{AB}dx^A\otimes d\left(dx^B\left(S_0\right)\right) && (3.4.21)\\
=& c^2d\left(c^2t^2+\delta_{AB}x^Ax^B\right)\otimes dt+c^2dt\otimes d\left(c^2t^2+\delta_{AB}x^Ax^B\right)\\
&-2c^2\delta_{AB}d\left(tx^A\right)\otimes dx^B-2c^2\delta_{AB}dx^A\otimes d\left(tx^B\right) && (3.4.22)\\
=& 2c^2\left(c^2tdt+\delta_{AB}x^Bdx^A\right)\otimes dt+2c^2dt\otimes\left(c^2tdt+\delta_{AB}x^Adx^B\right)\\
&-2c^2\delta_{AB}x^Adt\otimes dx^B-2c^2tdx^A\otimes dx^B-2c^2\delta_{AB}dx^A\otimes x^Bdt\\
&-2c^2\delta_{AB}dx^A\otimes tdx^B && (3.4.23)\\
=& -2c^2t\left(-c^2dt\otimes dt+\delta_{AB}dx^A\otimes dx^B\right) && (3.4.24)\\
=& -4c^2t\,\eta. && (3.4.25)
\end{aligned}
$$

The Lie derivative of $\tilde{\eta}$ with respect to S_0 can be proven to be $L_{S_0}\tilde{\eta}=4c^2t\,\tilde{\eta}$, therefore $L_{S_0}\left(\eta\otimes\tilde{\eta}\right)=0$ and condition (3.4.9) is satisfied. To find the transformation associated to S_0 we need to first solve the system of ODEs

$$
\begin{aligned}
0 =& \dot{x}(\lambda)\frac{\partial}{\partial x}+\dot{y}(\lambda)\frac{\partial}{\partial y}+\dot{z}(\lambda)\frac{\partial}{\partial z}+\dot{t}(\lambda)\frac{\partial}{\partial t}\\
&+2c^2t(\lambda)x(\lambda)\frac{\partial}{\partial x}+2c^2t(\lambda)y(\lambda)\frac{\partial}{\partial y}+2c^2t(\lambda)z(\lambda)\frac{\partial}{\partial z}\\
&+\left(c^2t(\lambda)^2+x(\lambda)^2+y(\lambda)^2+z(\lambda)^2\right)\frac{\partial}{\partial t}. && (3.4.26)
\end{aligned}
$$

This system has a unique solution for given initial conditions, which is used to construct the transformations as flows

$$h^{S_0}(\lambda,t,x,y,z)=\left(\frac{\left(-c^2t^2+x^2+y^2+z^2\right)\left(t-\lambda\left(-c^2t^2+x^2+y^2+z^2\right)\right)}{-c^2\left(t-\lambda\left(-c^2t^2+x^2+y^2+z^2\right)\right)^2+x^2+y^2+z^2},\right.$$
$$\frac{x\left(-c^2t^2+x^2+y^2+z^2\right)}{-c^2\left(t-\lambda\left(-c^2t^2+x^2+y^2+z^2\right)\right)^2+x^2+y^2+z^2},$$
$$\frac{y\left(-c^2t^2+x^2+y^2+z^2\right)}{-c^2\left(t-\lambda\left(-c^2t^2+x^2+y^2+z^2\right)\right)^2+x^2+y^2+z^2},$$
$$\left.\frac{z\left(-c^2t^2+x^2+y^2+z^2\right)}{-c^2\left(t-\lambda\left(-c^2t^2+x^2+y^2+z^2\right)\right)^2+x^2+y^2+z^2}\right). \tag{3.4.27}$$

The special conformal transformation in the x-direction has generator S_1 given by

$$S_1=\left(c^2t^2+x^2-y^2-z^2\right)\frac{\partial}{\partial x}+2xy\frac{\partial}{\partial y}+2xz\frac{\partial}{\partial z}+2tx\frac{\partial}{\partial t}. \tag{3.4.28}$$

The Lie derivative of the Minkowski metric η with respect to the vector field S_1 is computed as follows

$$L_{S_1}\eta = L_{S_1}\left(-c^2dt\otimes dt+\delta_{AB}dx^A\otimes dx^B\right) \tag{3.4.29}$$
$$= -c^2d\left(dt\left(S_1\right)\right)\otimes dt-c^2dt\otimes d\left(dt\left(S_1\right)\right)+\delta_{AB}d\left(dx^A\left(S_1\right)\right)\otimes dx^B + \delta_{AB}dx^A\otimes d\left(dx^B\left(S_1\right)\right) \tag{3.4.30}$$
$$= -c^2d\left(2tx\right)\otimes dt-c^2dt\otimes d\left(2tx\right)+\delta_{AB}d\left(2xx^A\right)\otimes dx^B + \delta_{AB}dx^A\otimes d\left(2xx^B\right)-d\left(\eta_{\mu\nu}x^\mu x^\nu\right)\otimes dx-dx\otimes d\left(\eta_{\mu\nu}x^\mu x^\nu\right) \tag{3.4.31}$$
$$= -4c^2xdt\otimes dt-2c^2tdx\otimes dt-2c^2tdt\otimes dx+4x\delta_{AB}dx^A\otimes dx^B + 2x^A\delta_{AB}dx\otimes dx^B+2x^B\delta_{AB}dx^A\otimes dx-2\eta_{\mu\nu}x^\mu dx^\nu\otimes dx - 2\eta_{\mu\nu}x^\nu dx\otimes dx^\mu \tag{3.4.32}$$
$$=4x\left(-c^2dt\otimes dt+\delta_{AB}dx^A\otimes dx^B\right) \tag{3.4.33}$$
$$=4x\,\eta. \tag{3.4.34}$$

The Lie derivative of $\tilde{\eta}$ with respect to S_1 can be proven to be $L_{S_1}\tilde{\eta}=-4x\,\tilde{\eta}$, therefore $L_{S_1}(\eta \otimes \tilde{\eta}) = 0$ and condition (3.4.9) is satisfied. To find the transformation associated to S_1 we need to first solve the system of ODEs

$$\begin{aligned} 0 =& \dot{x}(\lambda)\frac{\partial}{\partial x} + \dot{y}(\lambda)\frac{\partial}{\partial y} + \dot{z}(\lambda)\frac{\partial}{\partial z} + \dot{t}(\lambda)\frac{\partial}{\partial t} \\ &- \left(c^2 t(\lambda)^2 + x(\lambda)^2 - y(\lambda)^2 - z(\lambda)^2\right)\frac{\partial}{\partial x} \\ &- 2x(\lambda)y(\lambda)\frac{\partial}{\partial y} - 2x(\lambda)z(\lambda)\frac{\partial}{\partial z} - 2t(\lambda)x(\lambda)\frac{\partial}{\partial t}. \end{aligned} \tag{3.4.35}$$

This system has unique solution for given initial conditions, using this the transformations are built as a flow

$$\begin{aligned} h^{S_1}(\lambda, t, x, y, z) = \Bigg(& \frac{t\left(-c^2t^2 + x^2 + y^2 + z^2\right)}{\left(x - \lambda\left(-c^2t^2 + x^2 + y^2 + z^2\right)\right)^2 - c^2t^2 + y^2 + z^2}, \\ & \frac{\left(-c^2t^2 + x^2 + y^2 + z^2\right)\left(x - \lambda\left(-c^2t^2 + x^2 + y^2 + z^2\right)\right)}{\left(x - \lambda\left(-c^2t^2 + x^2 + y^2 + z^2\right)\right)^2 - c^2t^2 + y^2 + z^2}, \\ & \frac{y\left(-c^2t^2 + x^2 + y^2 + z^2\right)}{\left(x - \lambda\left(-c^2t^2 + x^2 + y^2 + z^2\right)\right)^2 - c^2t^2 + y^2 + z^2}, \\ & \frac{z\left(-c^2t^2 + x^2 + y^2 + z^2\right)}{\left(x - \lambda\left(-c^2t^2 + x^2 + y^2 + z^2\right)\right)^2 - c^2t^2 + y^2 + z^2}\Bigg). \end{aligned} \tag{3.4.36}$$

The special conformal transformation in the y-direction has generator S_2 given by

$$S_2 = 2xy\frac{\partial}{\partial x} + \left(c^2t^2 - x^2 + y^2 - z^2\right)\frac{\partial}{\partial y} + 2yz\frac{\partial}{\partial z} + 2ty\frac{\partial}{\partial t}. \tag{3.4.37}$$

The Lie derivative of the metric η with respect to the vector field S_2 is computed as follows

$$
\begin{aligned}
L_{S_2}\eta =& L_{S_2}\left(-c^2 dt\otimes dt + \delta_{AB} dx^A \otimes dx^B\right) && (3.4.38)\\
=& -c^2 d\left(dt\left(S_2\right)\right)\otimes dt - c^2 dt\otimes d\left(dt\left(S_2\right)\right) + \delta_{AB} d\left(dx^A\left(S_2\right)\right)\otimes dx^B + \delta_{AB} dx^A\otimes d\left(dx^B\left(S_2\right)\right) && (3.4.39)\\
=& -c^2 d\left(2ty\right)\otimes dt - c^2 dt\otimes d\left(2ty\right) + \delta_{AB} d\left(2yx^A\right)\otimes dx^B + \delta_{AB} dx^A \otimes d\left(2yx^B\right)\\
& - d\left(\eta_{\mu\nu}x^\mu x^\nu\right)\otimes dy - dy\otimes d\left(\eta_{\mu\nu}x^\mu x^\nu\right) && (3.4.40)\\
=& -4c^2 y dt\otimes dt - 2c^2 t dy\otimes dt - 2c^2 t dt\otimes dy + 4y\delta_{AB} dx^A\otimes dx^B\\
& + 2x^A\delta_{AB} dy\otimes dx^B + 2x^B\delta_{AB} dx^A\otimes dy - 2\eta_{\mu\nu}x^\mu dx^\nu\otimes dy - 2\eta_{\mu\nu}x^\nu dy\otimes dx^\mu && (3.4.41)\\
=& 4y\left(-c^2 dt\otimes dt + \delta_{AB} dx^A\otimes dx^B\right) && (3.4.42)\\
=& 4y\,\eta. && (3.4.43)
\end{aligned}
$$

The Lie derivative of $\tilde{\eta}$ with respect to S_2 can be proven to be $L_{S_2}\tilde{\eta} = -4y\,\tilde{\eta}$, therefore $L_{S_2}\left(\eta\otimes\tilde{\eta}\right) = 0$ and condition (3.4.9) is satisfied. To find the transformation associated to S_2 we need to first solve the system of ODEs

$$
\begin{aligned}
0 =& \dot{x}(\lambda)\frac{\partial}{\partial x} + \dot{y}(\lambda)\frac{\partial}{\partial y} + \dot{z}(\lambda)\frac{\partial}{\partial z} + \dot{t}(\lambda)\frac{\partial}{\partial t} && (3.4.44)\\
& - 2x(\lambda)y(\lambda)\frac{\partial}{\partial x} - \left(c^2 t(\lambda)^2 - x(\lambda)^2 + y(\lambda)^2 - z(\lambda)^2\right)\frac{\partial}{\partial y} - 2y(\lambda)z(\lambda)\frac{\partial}{\partial z} - 2t(\lambda)y(\lambda)\frac{\partial}{\partial t}. && (3.4.45)
\end{aligned}
$$

This system has a unique solution for given initial conditions. Using these solutions, the transformations are built as a flow

$$h^{S_2}(\lambda,t,x,y,z)=\left(\frac{t\left(-c^2t^2+x^2+y^2+z^2\right)}{\left(y-\lambda\left(-c^2t^2+x^2+y^2+z^2\right)\right)^2-c^2t^2+x^2+z^2},\right.$$
$$\frac{x\left(-c^2t^2+x^2+y^2+z^2\right)}{\left(y-\lambda\left(-c^2t^2+x^2+y^2+z^2\right)\right)^2-c^2t^2+x^2+z^2},$$
$$\frac{\left(-c^2t^2+x^2+y^2+z^2\right)\left(y-\lambda\left(-c^2t^2+x^2+y^2+z^2\right)\right)}{\left(y-\lambda\left(-c^2t^2+x^2+y^2+z^2\right)\right)^2-c^2t^2+x^2+z^2},$$
$$\left.\frac{z\left(-c^2t^2+x^2+y^2+z^2\right)}{\left(y-\lambda\left(-c^2t^2+x^2+y^2+z^2\right)\right)^2-c^2t^2+x^2+z^2}\right). \quad (3.4.46)$$

The special conformal transformation in the z-direction has generator S_3 given by

$$S_3=2xz\frac{\partial}{\partial x}+2yz\frac{\partial}{\partial y}+\left(c^2t^2-x^2-y^2+z^2\right)\frac{\partial}{\partial z}+2tz\frac{\partial}{\partial t}. \quad (3.4.47)$$

The Lie derivative of the metric η with respect to the vector field S_3 is computed as follows

$$
\begin{aligned}
L_{S_1}\eta =& L_{S_3}\left(-c^2dt\otimes dt+\delta_{AB}dx^A\otimes dx^B\right) && (3.4.48)\\
=&-c^2d\left(dt\left(S_3\right)\right)\otimes dt-c^2dt\otimes d\left(dt\left(S_3\right)\right)+\delta_{AB}d\left(dx^A\left(S_3\right)\right)\otimes dx^B\\
&+\delta_{AB}dx^A\otimes d\left(dx^B\left(S_3\right)\right) && (3.4.49)\\
=&-c^2d\left(2tz\right)\otimes dt-c^2dt\otimes d\left(2tz\right)+\delta_{AB}d\left(2zx^A\right)\otimes dx^B\\
&+\delta_{AB}dx^A\otimes d\left(2zx^B\right)-d\left(\eta_{\mu\nu}x^\mu x^\nu\right)\otimes dz-dz\otimes d\left(\eta_{\mu\nu}x^\mu x^\nu\right) && (3.4.50)\\
=&-4c^2zdt\otimes dt-2c^2tdz\otimes dt-2c^2tdt\otimes dz+4z\delta_{AB}dx^A\otimes dx^B\\
&+2x^A\delta_{AB}dz\otimes dx^B+2x^B\delta_{AB}dx^A\otimes dz-2\eta_{\mu\nu}x^\mu dx^\nu\otimes dz\\
&-2\eta_{\mu\nu}x^\nu dz\otimes dx^\mu && (3.4.51)\\
=&4z\left(-c^2dt\otimes dt+\delta_{AB}dx^A\otimes dx^B\right) && (3.4.52)\\
=&4z\,\eta. && (3.4.53)
\end{aligned}
$$

The Lie derivative of $\tilde{\eta}$ with respect to S_3 can be proven to be $L_{S_3}\tilde{\eta}=-4z\,\tilde{\eta}$, therefore $L_{S_3}\left(\eta\otimes\tilde{\eta}\right)=0$ and condition (3.4.9) is satisfied. To find the transformation associated to S_3 we need to first solve the system of ODEs

$$\begin{aligned} 0 =&\dot{x}(\lambda)\frac{\partial}{\partial x} + \dot{y}(\lambda)\frac{\partial}{\partial y} + \dot{z}(\lambda)\frac{\partial}{\partial z} + \dot{t}(\lambda)\frac{\partial}{\partial t} \\ &- 2x(\lambda)z(\lambda)\frac{\partial}{\partial x} - 2t(\lambda)z(\lambda)\frac{\partial}{\partial t} \\ &- 2y(\lambda)z(\lambda)\frac{\partial}{\partial y} - \left(c^2t(\lambda)^2 - x(\lambda)^2 - y(\lambda)^2 + z(\lambda)^2\right)\frac{\partial}{\partial z}. \end{aligned} \tag{3.4.54}$$

This system has a unique solution for given initial conditions, which are used to construct the transformation as a flow

$$\begin{aligned} h^{S_3}(\lambda, t, x, y, z) = \Bigg(&\frac{t\left(-c^2t^2 + x^2 + y^2 + z^2\right)}{\left(z - \lambda\left(-c^2t^2 + x^2 + y^2 + z^2\right)\right)^2 - c^2t^2 + x^2 + y^2}, \\ &\frac{x\left(-c^2t^2 + x^2 + y^2 + z^2\right)}{\left(z - \lambda\left(-c^2t^2 + x^2 + y^2 + z^2\right)\right)^2 - c^2t^2 + x^2 + y^2}, \\ &\frac{y\left(-c^2t^2 + x^2 + y^2 + z^2\right)}{\left(z - \lambda\left(-c^2t^2 + x^2 + y^2 + z^2\right)\right)^2 - c^2t^2 + x^2 + y^2}, \\ &\frac{\left(-c^2t^2 + x^2 + y^2 + z^2\right)\left(z - \lambda\left(-c^2t^2 + x^2 + y^2 + z^2\right)\right)}{\left(z - \lambda\left(-c^2t^2 + x^2 + y^2 + z^2\right)\right)^2 - c^2t^2 + x^2 + y^2}\Bigg). \end{aligned} \tag{3.4.55}$$

The conformal group consists then of space-time translations, space rotations, boosts, space-time dilations and special conformal transformations. It was shown by Coleman-Mandula in Coleman and Mandula (1967) that this is the most general space-time symmetry group of a non-trivial, relativistic field theory and has a strong presence in theoretical physics.

3.4.1 Algebra

The algebra of the group ISO $(3,1)$ is obtained by taking the differential-geometric commutator of all generators of the group and is given by

$$[P_A, P_B] = 0 \qquad [P_A, H] = 0 \qquad [P_A, J_B] = \epsilon_{ABC} J_C \tag{3.4.56}$$

$$[P_A, B_B] = \delta_{AB} H \qquad [H, J_A] = 0 \qquad [H, B_A] = c P_A \tag{3.4.57}$$

$$[J_A, J_B] = \epsilon_{ABC} J_C \qquad [J_A, B_B] = \epsilon_{ABC} B_C \qquad [H, S_A] = 2 B_A \tag{3.4.58}$$

$$[H, S_0] = 2cD \qquad [P_A, S_B] = 2\delta_{AB} D + \epsilon_{ABC} J_C \qquad [P_A, S_0] = -2c B_A \tag{3.4.59}$$

$$[J_A, S_B] = \epsilon_{ABC} S_C \qquad [J_A, S_0] = 0 \qquad [B_A, S_B] = -\frac{1}{c} \delta_{AB} S_0 \tag{3.4.60}$$

$$[B_A, S_0] = -c B_A \qquad [P_A, D] = P_A \qquad [H, D] = -H \tag{3.4.61}$$

$$[J_A, D] = 0 \qquad [B_A, D] = 0 \qquad [D, S_A] = S_A \tag{3.4.62}$$

$$[D, S_0] = S_0. \tag{3.4.63}$$

Chapter 4

Carroll group and Carrollian algebra

The Carroll group was first described by Levy-Leblond in 1968 in Bacry and Levy-Leblond (1968) in an effort to categorize all possible kinematic groups that allow for boosts, rotations, translations and have some notion of causality. The Carrollian limit is also called the ultra-relativistic limit, in which the speed of light is taken to zero. This causes the collapse of all causal cones into causal lines, meaning every point can only causally affect itself. Because of this, it was first believed Carroll-invariant field theories were all static, but there have been models found to admit dynamical solutions in the presence of interactions.

The Carrollian Lie algebra was first obtained as a contraction of the Poincaré algebra as the limiting case of taking the speed of light to zero and was, rightfully so, ignored since there was no physical reason to be interested in it.

Now, of course, we have reasons to care.

Since then, Carrollian geometry has found its footing in theoretical physics in the gravity and cosmological side. From effective physics at null infinity Herfray (2022) to dark matter studies Avila et al. (2023), Carroll has become a part of the landscape of relevant groups in our discipline.

Study of Carrollian limits requires a background on Carrollian geometry, which is briefly presented here. We start by defining some structure. A Carroll manifold is a quadruple (C, g, ξ, ∇), where Duval et al. (2014c)

- C is a $(d+1)$-smooth manifold.

- g is a rank 2 degenerate metric tensor field.
- ξ is a nowhere vanishing complete vector field which spans $\ker(g)$.
- ∇ is a symmetric affine connection that parallel transports both g and ξ.

Notice that there is a significant difference between a Lorentzian manifold and a Carrollian one. Namely, there is not a non-degenerate metric tensor field[1] in a Carrollian manifold. An immediate consequence of this is there is mostly no metric induced structure, no pseudo-inner products to be had, no volume form constructed from the metric.

A Carroll group is the group of automorphisms of a Carroll structure.

4.1 Flat Carrollian structure

The standard flat Carroll structure is given by the choice

$$C^{d+1} = \mathbb{R} \times \mathbb{R}^d \qquad g = \delta_{AB} dx^A \otimes dx^B \qquad \xi = \frac{\partial}{\partial s} \qquad \Gamma^i_{jk} = 0.$$

This spatial Carroll metric can also be obtained from the Minkowski metric $dS^2 = -dx^0 \otimes dx^0 + \delta_{AB} dx^A \otimes dx^B$ by choosing $s = Cx^0$ and taking $C \to \infty$. These choices may seem arbitrary but are in fact a consequence of them coming from Minkowski spacetime. Indeed, the reason why the underlying manifold is a power of the real numbers and the connection is set to zero is the same.

4.2 Flat Carroll group action

The Carroll group is formed by the set of automorphisms that preserve the Carrollian structure. Let $a : C^{d+1} \to C^{d+1}$ be one such a map, then we have

$$a^* g = g \qquad a_* \xi = \xi \qquad a^* \nabla = \nabla. \tag{4.2.1}$$

[1]Although there exists a killing form in $(2+1)$ dimensions Matulich et al. (2019).

The set of all transformations that leave the flat Carrollian structure invariant is the flat Carroll group $Carr\,(d+1)$. This is

$$Carr\,(d+1) = \left\{ a \in \text{End}\left(C^{d+1}\right) \middle|\, a^* g = g \wedge a_* \xi = \xi \wedge a^* \nabla = \nabla \right\}. \tag{4.2.2}$$

This set is conformed by spatial rotations, time translations, space translations and time boosts.

The bilinear function g is only degenerate in the full Carrollian manifold. If we were to consider the restriction to the spatial part we would be looking at the standard inner product in $\mathbb{R}^d$. It follows we have a well-defined notion of rotations in this submanifold given by the group orthogonal group $O(d)$.

Time translations are characterized by the additive group $(\mathbb{R}, +)$. Likewise, spatial translations are characterized by $\left(\mathbb{R}^d, +\right)$.

Time boosts are also formed by the additive group $\left(\mathbb{R}^d, +\right)$ but their action comes in a slightly more complicated way, namely, a semi-direct product.

Putting it all together, the flat Carroll group can be written as

$$C^{d+1} = \left(\mathbb{R} \oplus \mathbb{R}^d\right) \rtimes O\,(d)\,. \tag{4.2.3}$$

Given our choices, representatives of $Carr(d+1)$ and C^{d+1} are of the form

$$a = \begin{pmatrix} R & 0 & \mathbf{c} \\ -\mathbf{b}^T R & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \in Carr(d+1) \qquad x = \begin{pmatrix} s \\ \mathbf{x} \end{pmatrix} \in C^{d+1}, \tag{4.2.4}$$

where $R \in O(d)$ is a d-dimensional orthogonal matrix, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^d$ are both d-dimensional real vectors and $f \in \mathbb{R}$ is a real number. Which allows us to define the action

$$\triangleright_C : Carr(d+1) \times C^{d+1} \longrightarrow C^{d+1} \tag{4.2.5}$$

$$(a, x) \longrightarrow a \triangleright_C x := \begin{bmatrix} s - \mathbf{b}^T R\mathbf{x} + f \\ R\mathbf{x} + \mathbf{c} \end{bmatrix}, \tag{4.2.6}$$

which is just the matrix multiplication of a and x.

4.3 Carrollian Lie algebra

With this in mind, it's possible to define its Lie algebra $\mathfrak{carr}(d+1)$, which is composed of vector field of the form

$$X = \left(\omega^A{}_B x^B + \gamma^A\right) \frac{\partial}{\partial x^A} + \left(\varphi - \beta_A x^A\right) \frac{\partial}{\partial s}, \tag{4.3.1}$$

where $\omega \in \mathfrak{so}(d)$, $\beta, \gamma \in \mathbb{R}^d$ and $\varphi \in \mathbb{R}$. Vector fields $X \in \mathfrak{carr}\,(d+1)$ satisfy the infinitesimal version of invariance conditions shown in (4.2.1)

$$L_X g = 0 \qquad L_X \xi = 0 \qquad L_X \nabla = 0. \tag{4.3.2}$$

These conditions allows us to define the flat[2] Carrollian lie algebra $\mathfrak{carr}\,(d+1)$ as

$$\mathfrak{carr}\,(d+1) := \left\{ X \in \Gamma\left(TC^{d+1}\right) \middle|\, L_X g = 0 \wedge L_X \xi = 0 \wedge L_X \nabla = 0 \right\}. \tag{4.3.3}$$

Of course, one can consider each kind of transformation separately

$$P_A = \frac{\partial}{\partial x^A} \qquad J_A = \epsilon_{ABC} x_B \frac{\partial}{\partial x^C} \qquad K_A = x_A \frac{\partial}{\partial s} \qquad P_0 = \frac{\partial}{\partial s} \tag{4.3.4}$$

[2]The insistence on *flat* is to distinguish it from the conformal case, in which the requirement placed on the connection is dropped and the remaining two conditions are modified.

and construct the algebra by taking the differential-geometric commutator of these vector fields

$$[\cdot,\cdot] : \Gamma\left(TC^{d+1}\right) \times \Gamma\left(TC^{d+1}\right) \longrightarrow \Gamma\left(TC^{d+1}\right) \tag{4.3.5}$$
$$(A,B) \longrightarrow [A,B]. \tag{4.3.6}$$

This is

$$[J_A, J_B] = \epsilon_{ABC} J_C \qquad [J_A, K_B] = \epsilon_{ABC} K_C \qquad [K_A, K_B] = 0 \tag{4.3.7}$$
$$[J_A, P_B] = \epsilon_{ABC} P_C \qquad [K_A, P_B] = \delta_{AB} P_0 \qquad [J_A, P_0] = 0 \tag{4.3.8}$$
$$[K_A, P_0] = 0 \qquad [P_A, P_B] = 0 \qquad [P_A, P_0] = 0. \tag{4.3.9}$$

This can also be obtained as contraction from the Poincare algebra, as was done in the original paper Bacry and Levy-Leblond (1968).

Reconstruction of the symmetries by using their generators is possible by the usual method, which has been done for Lorentz symmetry in 3.2 and in this case reproduces the action $\triangleright_C$ defined in (4.2.6). Explicit reconstruction of Carrollian symmetries is done for electrodynamics in a following chapter.

4.4 Conformal extension

Carrollian limits of Maxwell and ModMax theory have symmetries belonging to the conformal Carroll group of level two. For this reason we must employ some time talking about conformal extensions of the flat Carroll group $Carr\,(d+1)$. Flatness as a requirement is dropped for the conformal extensions[3], which means time translations and Carrollian boosts can be condensed in a single $\left(C^\infty\left(\mathbb{R}^d\right),+\right)$ additive group. Let $f \in C^\infty\left(\mathbb{R}^d\right)$, then super-translations

[3]The paper Duval et al. (2014a) also makes the distinction between strong Carroll structure $\left(C^{d+1}, g, \xi, \nabla\right)$ and weak Carroll structure $\left(C^{d+1}, g, \xi\right)$, with the infinite dimensional super-translations being a part of the endomorphisms of weak Carroll structure.

$$s \longrightarrow s + f(x, y, z) \tag{4.4.1}$$

are allowed as part of any conformal extension of $Carr\,(d+1)$. The flat Carroll group admits various conformal extensions, which appear in different cases of Carrollian limits of relativistic conformal field theories. These extensions are categorized by a natural number k. Let us then define the Conformal Carroll group of level k $CCarr_k\,(d+1)$ to be the set

$$CCarr_k\,(d+1) := \left\{ a \in \mathrm{End}\left(C^{d+1}\right) \middle| \, a^* \left(g \otimes \xi^{\otimes k}\right) = g \otimes \xi^{\otimes k} \right\}, \tag{4.4.2}$$

where $k \in \mathbb{N}_0$ is a natural number and $\xi^{\otimes k}$ is the k-th tensor power of the vector field ξ

$$\xi^{\otimes k} := \bigotimes_{n=0}^{k} \xi. \tag{4.4.3}$$

Requirement (4.4.2) can be put into differential form, which allows to define the conformal Carrollian Lie algebra of level k, denoted by $\mathfrak{ccarr}_k\,(d+1)$.

First let the group element $a = \exp(\lambda X)$ be a 1-parameter subgroup generated by $X \in \mathfrak{ccarr}_k\,(d+1)$, then we can construct a Lie derivative by using a differential quotient with the group parameter λ

$$\exp(\lambda X)^* \left(g \otimes \xi^{\otimes k}\right) - g \otimes \xi^{\otimes k} = 0 \tag{4.4.4}$$

$$\lim_{\lambda \to 0} \frac{\exp(\lambda X)^* \left(g \otimes \xi^{\otimes k}\right) - g \otimes \xi^{\otimes k}}{\lambda} = 0 \tag{4.4.5}$$

$$L_X \left(g \otimes \xi^{\otimes k}\right) = 0 \tag{4.4.6}$$

Then the desired definition of the conformal Carrollian Lie algebra of level k is

$$\mathfrak{ccarr}_k\left(d+1\right) := \left\{X \in \Gamma\left(TC^{d+1}\right) \middle| \, L_X\left(g \otimes \xi^{\otimes k}\right) = 0\right\} \tag{4.4.7}$$

Sufficient and necessary condition for this to happen is[4]

$$L_X g = \Omega g \qquad\qquad L_X \xi = -\frac{\Omega}{k}\xi \tag{4.4.8}$$

Where $\Omega : C^{d+1} \longrightarrow \mathbb{R}$ is an arbitrary real-valued section over the space C^{d+1}. Were this not the case, there would be additional terms in the Lie derivative $L_X\left(g \otimes \xi^{\otimes k}\right)$. Further analysis of conformal Carrollian groups of level $k=2$ are done when discussing the symmetries of Carrollian limits of Maxwell theory, other cases are beyond the scope of this work.

[4]These conditions are more useful for actual computations, which will be seen further along the road.

Chapter 5

Carrollian limits of a scalar field

The simplest $(3+1)$-Lorentzian theory in which we can take the Carrollian limit and therefore use as an example for the procedure of the work to come is that of the scalar field. We start by writing the action of the free scalar field

$$S\left[\phi, d\phi\right] = -\int_{\Omega} \frac{1}{2}\left\langle d\phi, d\phi\right\rangle d^4x \tag{5.0.1}$$

$$= \int_{\Omega}\left[\frac{1}{2c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\nabla\phi\cdot\nabla\phi\right] d^4x, \tag{5.0.2}$$

where $\Omega \subseteq \mathbb{R}^4$ is an open submanifold of Minkowski spacetime. Arriving at the equations of motion for this field theory is standard practice and can be seen, for example, in Schwartz (2014). In this case, a procedure inspired by Goldstein et al. (2002) will be used. We start by considering a variation characterized by a one-parameter family of scalar fields

$$\phi(\alpha) := \phi + \alpha a, \tag{5.0.3}$$

where $\alpha \in \mathbb{R}$ and a is a C^1 Lebesgue integrable scalar field and we assume that the action gets an extreme value at $\phi(0)$. This way, the action in (5.0.1) becomes $S\left[\phi(0), d\phi(0)\right]$, where

$$
\begin{aligned}
S\left[\phi(\alpha), d\phi(\alpha)\right] &= -\int_{\Omega} \langle d\phi(\alpha), d\phi(\alpha)\rangle \, d^4x & (5.0.4)\\
&= -\int_{\Omega} \left[\frac{1}{2}\langle d\phi, d\phi\rangle + \alpha \langle d\phi, da\rangle + \frac{1}{2}\alpha^2 \langle da, da\rangle\right] d^4x. & (5.0.5)
\end{aligned}
$$

To be able to carry on it is important to integrate by parts the middle term

$$
\begin{aligned}
-\int_{\Omega} \langle d\phi, da\rangle \, d^4x &= \int_{\Omega} \star d\phi \wedge da + d \star d\phi \wedge a - d \star d\phi \wedge a & (5.0.6)\\
&= \int_{\partial\Omega} \star d\phi \wedge a - \int_{\Omega} d \star d\phi \wedge a. & (5.0.7)
\end{aligned}
$$

Using this we can construct the functional derivative as we would a real derivative

$$
\begin{aligned}
\frac{dS[\phi, d\phi]}{d\alpha}(0) :&= \lim_{\alpha \to 0} \frac{S[\phi(\alpha), d\phi(\alpha)] - S[\phi(0), d\phi(0)]}{\alpha} & (5.0.8)\\
&= \lim_{\alpha \to 0} \int_{\mathbb{R}^4} \left[\langle \star d \star d\phi, a\rangle - \frac{1}{2}\alpha \langle da, da\rangle\right] d^4x & (5.0.9)\\
&= \int_{\mathbb{R}^4} \langle \star d \star d\phi, a\rangle \, d^4x, & (5.0.10)
\end{aligned}
$$

and since the pseudo-inner product is non-degenerate[1] it follows that

$$
\star d \star d\phi = 0. \qquad (5.0.11)
$$

This, of course, is just the wave equation for the scalar field

$$
\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = 0. \qquad (5.0.12)
$$

[1] Although in this case it is a little obscene to talk about pseudo-inner products since both $\star d \star d\phi$ and a are scalars and $\langle \star d \star d\phi, a\rangle = a \star d \star d\phi$. However, this serves to illustrate the Maxwell case.

Now[2], it is important to remark here the role played by the pseudo-inner-product of Lorentzian geometry.

1. Both the Hodge star operator and pseudo-inner products $\langle \cdot, \cdot \rangle$ are explicitly constructed in terms of the metric, which implies the of the equation of motion is metric-dependent.
2. The volume measure is also constructed to be metric-compatible.

This is an integral part of Lagrangian descriptions of fields. In fact, in classical mechanics it is usually overlooked that you only can construct Lagrangians because you can take two velocity vectors and map them into kinetic energy. This is, of course, not the case in neither Carrollian nor in Galilean geometry where there is not a non-degenerate bilinear form.

At this point, you could multiply (5.0.12) by c^2 or make the transition to Carrollian units and taking the limit $c \to 0$ or $C \to \infty$ to the same effect. Personally, I'll do the latter for consistency

$$\frac{\partial^2 \phi}{\partial s^2} = 0. \tag{5.0.13}$$

This is Carroll invariant because neither $\frac{\partial}{\partial s}$ nor ϕ transform under Carroll. And that's it, right? The Carrollian limit of the scalar field. Well, no. The free scalar field admits two non-equivalent Carrollian limits, the so-called electric we just obtained and the so-called magnetic one which can be obtained via Hamiltonian formalism.

5.1 At the level of the Hamiltonian

Both electric and magnetic limits can be independently constructed from the Hamiltonian formalism of free scalar field theory. The magnetic limit is obtained by taking the limit $c \to 0$ in the action written in canonical variables. The electric limit is obtained by doing the same after a convenient field reparametrization.

[2] Dear reader, if you're thinking I could've just used a coordinate description and be done with it you'd be absolutely correct. However, you can't stop me.

We start by building the action in canonical variables, for doing so we must define the canonical momentum density $\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{1}{c^2}\frac{\partial \phi}{\partial t}$. We arrive to the Hamiltonian action principle after performing the Legendre transformation

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}\dot{\phi} - \mathcal{L} = \frac{c^2}{2}\pi^2 + \frac{1}{2}\nabla\phi \cdot \nabla\phi. \tag{5.1.1}$$

Replacing this in the action allows us to separate the degrees of freedom into the scalar field and its canonically conjugate momentum π

$$\begin{aligned} S[\pi, \phi] &= \int_{\mathbb{R}^4} d^4x \left[\pi\dot{\phi} - \mathcal{H}\right] && (5.1.2)\\ &= \int_{\mathbb{R}^4} d^4x \left[\frac{c^2}{2}\pi^2 - \frac{1}{2}\nabla\phi \cdot \nabla\phi\right]. && (5.1.3) \end{aligned}$$

The equations of motion in the Hamiltonian formalism for the scalar field are

$$\begin{aligned} \dot{\pi} &= -\frac{\partial \mathcal{H}}{\partial \phi} + \frac{\partial}{\partial x_i}\left(\frac{\partial \mathcal{H}}{\partial(\partial_i \phi)}\right) & \dot{\phi} &= \frac{\partial \mathcal{H}}{\partial \pi} - \frac{\partial}{\partial x_i}\left(\frac{\partial \mathcal{H}}{\partial(\partial_i \pi)}\right) && (5.1.4)\\ &= \nabla^2\phi & &= \pi. && (5.1.5) \end{aligned}$$

By combining these two equations the wave equation is recovered.

5.1.1 Magnetic limit of the scalar free field

As was previously stated, the magnetic limit of free scalar field theory is recovered by taking the limit $c \to 0$ in the action written in term of canonical variables (ϕ, π), as was shown in Henneaux and Salgado-Rebolledo (2021)

$$S_M[\pi, \phi] = -\int_{\mathbb{R}^4} d^4x\, \mathcal{H}^M, \tag{5.1.6}$$

with Hamiltonian density given by

$$\mathcal{H}^M = \nabla\phi \cdot \nabla\phi. \tag{5.1.7}$$

Computation of the equations of motion gives as result

$$\dot{\pi} = \nabla^2\phi \qquad\qquad \dot{\phi} = 0. \tag{5.1.8}$$

An important thing to consider in this case is that for any temporal slice, solutions of the EOM will satisfy Laplace's equation. In the previous chapter it was said different kind of conformal field theories' Carrollian limits may have different k-level of Carrollian conformal symmetry. This is an example of such cases.

The following was obtained by the Lie point symmetry method, by considering the space $(\mathcal{F}_m, \text{п}, C^{3+1})$ with independent variables $(s, \mathbf{x})$ and dependent variables (ϕ, π) with projection map

$$\text{п} : \mathcal{F}_m \longrightarrow C^{3+1} \tag{5.1.9}$$

$$(s, \mathbf{x}, \phi, \pi) \longrightarrow \text{п}\,(s, \mathbf{x}, \phi, \pi) = (s, \mathbf{x})\,. \tag{5.1.10}$$

These equations of motion are constructed from the extended tangent space of $\mathcal{F}_m$[3] and their symmetries include time translations, spatial translations[4], spatial rotations, spatial dilations, special conformal transformations, time dilations, field dilations and an infinite sector that comes from there not being spatial derivatives of the conjugate momentum. These symmetries were obtained by considering the pair (5.1.8) as they stand.

Time translations have as generator the vector field $\mathcal{P}_0 \in \Gamma\,(T\mathcal{F}_m)$ given by

[3]Strictly speaking, the necessary math to properly talk about this machinery is that of jet bundles. However, that's beyond the scope of this work.

[4]Carrollian boosts were not obtained by this method although we know they are a symmetry of this system of equations.

$$\mathcal{P}_0 = \frac{\partial}{\partial s}. \tag{5.1.11}$$

The transformation for each generator $\mathcal{X} \in \Gamma\left(T\mathcal{F}_m\right)$ is built by the usual method. Let $p \in \mathcal{F}_m = (s_0, \mathbf{x}_0, \phi_0, \pi_0)$ be a point to serve as initial conditions for the system of ordinary differential equations $\dot{\gamma}^{\mathcal{X}}(\lambda) = \mathcal{X}_{\gamma^{\mathcal{X}}(\lambda)}$, with curve $\gamma^{\mathcal{X}} : \mathbb{R} \longrightarrow \mathcal{F}_m$. A solution to this system with initial conditions $\gamma^{\mathcal{X}}(0) = p$ is denoted by $\gamma_p^{\mathcal{X}}(\lambda)$. Symmetry transformations are then built by using the flows

$$h^{\mathcal{X}} : \mathbb{R} \times \mathcal{F}_m \longrightarrow \mathcal{F}_m \tag{5.1.12}$$

$$(\lambda, f) \longrightarrow h^{\mathcal{X}}\left(\lambda, f\right) = \gamma_f^{\mathcal{X}}(\lambda). \tag{5.1.13}$$

By doing this, time translations are recovered as a flow

$$h^{\mathcal{P}_0} : \mathbb{R} \times \mathcal{F}_m \longrightarrow \mathcal{F}_m \tag{5.1.14}$$

$$(\lambda, s, \mathbf{x}, \phi, \pi) \longrightarrow h^{\mathcal{P}_0}\left(\lambda, s, \mathbf{x}, \phi, \pi\right) = \left(s + \lambda, \mathbf{x}, \phi, \pi\right). \tag{5.1.15}$$

Spatial translations have as generator the vector field $\mathcal{P}_A \in \Gamma\left(T\mathcal{F}_m\right)$ given by

$$\mathcal{P}_A = \frac{\partial}{\partial x^A}. \tag{5.1.16}$$

Spatial translations are recovered as flows

$$h^{\mathcal{P}_1}: \mathbb{R} \times \mathcal{F}_m \longrightarrow \mathcal{F}_m \tag{5.1.17}$$
$$(\lambda, s, \mathbf{x}, \phi, \pi) \longrightarrow h^{\mathcal{P}_1}(\lambda, s, \mathbf{x}, \phi, \pi) = (s, x+\lambda, y, z, \phi, \pi) \tag{5.1.18}$$
$$h^{\mathcal{P}_2}: \mathbb{R} \times \mathcal{F}_m \longrightarrow \mathcal{F}_m \tag{5.1.19}$$
$$(\lambda, s, \mathbf{x}, \phi, \pi) \longrightarrow h^{\mathcal{P}_2}(\lambda, s, \mathbf{x}, \phi, \pi) = (s, x, y+\lambda, z, \phi, \pi) \tag{5.1.20}$$
$$h^{\mathcal{P}_3}: \mathbb{R} \times \mathcal{F}_m \longrightarrow \mathcal{F}_m \tag{5.1.21}$$
$$(\lambda, s, \mathbf{x}, \phi, \pi) \longrightarrow h^{\mathcal{P}_3}(\lambda, s, \mathbf{x}, \phi, \pi) = (s, x, y, z+\lambda, \phi, \pi)\,. \tag{5.1.22}$$

Spatial rotations have as generator the vector field $\mathcal{J}_A \in \Gamma(T\mathcal{F}_m)$ given by

$$\mathcal{J}_A = \epsilon_{ABC} x^B \frac{\partial}{\partial x^C}. \tag{5.1.23}$$

Spatial rotations are recovered as a flow

$$h^{\mathcal{J}_A}: \mathbb{R} \times \mathcal{F}_m \longrightarrow \mathcal{F}_m \tag{5.1.24}$$
$$(\lambda, s, \mathbf{x}, \phi, \pi) \longrightarrow h^{\mathcal{J}_A}(\lambda, s, \mathbf{x}, \phi, \pi) = (s, R_A(\lambda)\mathbf{x}, \phi, \pi)\,, \tag{5.1.25}$$

where $R_A(\lambda) \in \mathrm{SO}(3)$ is the A-th rotation matrix of angle λ. Details of this are given a further in the text so it is not worth it to have them here.

Spatial dilations have as generator the vector field $\mathcal{D} \in \Gamma(T\mathcal{F}_m)$ given by

$$\mathcal{D} = x^A \frac{\partial}{\partial x^A} + 2\phi \frac{\partial}{\partial \phi}. \tag{5.1.26}$$

Space dilations are recovered as a flow

$$h^{\mathcal{D}}: \mathbb{R} \times \mathcal{F}_m \longrightarrow \mathcal{F}_m \tag{5.1.27}$$
$$(\lambda, s, \mathbf{x}, \phi, \pi) \longrightarrow h^{\mathcal{D}}(\lambda, s, \mathbf{x}, \phi, \pi) = \left(s, e^{\lambda}\mathbf{x}, e^{2\lambda}\phi, \pi\right). \tag{5.1.28}$$

Time dilations have as generator the vector field $\mathcal{Q} \in \Gamma\left(T\mathcal{F}_m\right)$ given by

$$\mathcal{Q} = s\frac{\partial}{\partial s} - \phi\frac{\partial}{\partial \phi}. \tag{5.1.29}$$

Time dilations are recovered as a flow

$$h^{\mathcal{Q}} : \mathbb{R} \times \mathcal{F}_m \longrightarrow \mathcal{F}_m \tag{5.1.30}$$

$$(\lambda, s, \mathbf{x}, \phi, \pi) \longrightarrow h^{\mathcal{Q}}\left(\lambda, s, \mathbf{x}, \phi, \pi\right) = \left(e^{\lambda}s, \mathbf{x}, e^{-\lambda}\phi, \pi\right). \tag{5.1.31}$$

Special conformal transformations have as generator the vector field $\mathcal{S}_A \in \Gamma\left(T\mathcal{F}_m\right)$ given by

$$\mathcal{S}_A = 2x_A x^B \frac{\partial}{\partial x^B} - x_B x^B \frac{\partial}{\partial x^A} - 5x_A \pi \frac{\partial}{\partial \pi} - x_A \frac{\partial}{\partial \phi}. \tag{5.1.32}$$

The flow that serves as action of each special conformal transformation is

$$h^{\mathcal{S}_A} : \mathbb{R} \times \mathcal{F}_m \longrightarrow \mathcal{F}_m \tag{5.1.33}$$

$$(\lambda, s, \mathbf{x}, \phi, \pi) \longrightarrow h^{\mathcal{S}_A}\left(\lambda, s, \mathbf{x}, \phi, \pi\right) = \Big(s, \omega_A(\lambda)\left(\mathbf{x} - \text{и}_A(\lambda)\mathbf{x}\cdot\mathbf{x}\right), \Omega_A(\lambda)^{1/2}\phi, \Omega_1(\lambda)^{5/2}\pi\Big). \tag{5.1.34}$$

Where $\text{и}_A : \mathbb{R} \to \mathbb{R}^3$ with $\text{и}_1(\lambda) = (\lambda, 0, 0)$, $\text{и}_2(\lambda) = (0, \lambda, 0)$, $\text{и}_3(\lambda) = (0, 0, \lambda)$ and

$$\Omega_1(\lambda) = \left((\lambda x - 1)^2 + (\lambda y)^2 + (\lambda z)^2\right) \quad \omega_1(\lambda) = \frac{(x^2+y^2+z^2)}{\left(x - \lambda\left(x^2+y^2+z^2\right)\right)^2 + y^2 + z^2} \tag{5.1.35}$$

$$\Omega_2(\lambda) = \left((\lambda x)^2 + (\lambda y - 1)^2 + (\lambda z)^2\right) \quad \omega_2(\lambda) = \frac{(x^2+y^2+z^2)}{\left(y - \lambda\left(x^2+y^2+z^2\right)\right)^2 + x^2 + z^2} \tag{5.1.36}$$

$$\Omega_3(\lambda) = \left((\lambda x)^2 + (\lambda y)^2 + (\lambda z - 1)^2\right) \quad \omega_3(\lambda) = \frac{(x^2+y^2+z^2)}{\left(z - \lambda\left(x^2+y^2+z^2\right)\right)^2 + x^2 + y^2}. \tag{5.1.37}$$

Remark: these are not the special conformal transformations that appear in Carrollian electrodynamics, as can be seen in 7.1.3.

Field dilations have as generator the vector field $\mathcal{W} \in \Gamma\left(T\mathcal{F}_m\right)$ given by

$$\mathcal{W} = \phi\frac{\partial}{\partial\phi} + \pi\frac{\partial}{\partial\pi}. \tag{5.1.38}$$

The symmetries associated to this generator are recovered as a flow

$$h^{\mathcal{W}} : \mathbb{R} \times \mathcal{F}_m \longrightarrow \mathcal{F}_m \tag{5.1.39}$$

$$(\lambda, s, \mathbf{x}, \phi, \pi) \longrightarrow h^{\mathcal{W}}\left(\lambda, s, \mathbf{x}, \phi, \pi\right) = \left(s, \mathbf{x}, e^{\lambda}\phi, e^{\lambda}\pi\right). \tag{5.1.40}$$

The remaining symmetry has as generator the vector field $\mathcal{Y}_{abcd} \in \Gamma\left(T\mathcal{F}_m\right)$, where $a, b, c, d \in \mathbb{N}_0$ are natural numbers. This acts adding powers of space coordinates and scalar field ϕ to the canonical momentum π. These generators are given by

$$\mathcal{Y}_{abcd} = x^a y^b z^c \phi^d \frac{\partial}{\partial\pi}. \tag{5.1.41}$$

Symmetry transformations for each of them are recovered as a flow

$$h^{\mathcal{Y}_{abcd}} : \mathbb{R} \times \mathcal{F}_m \longrightarrow \mathcal{F}_m \tag{5.1.42}$$

$$(\lambda, s, \mathbf{x}, \phi, \pi) \longrightarrow h^{\mathcal{Y}_{abcd}} (\lambda, s, \mathbf{x}, \phi, \pi) = h^{\mathcal{Y}_{abcd}} \left(s, \mathbf{x}, \phi, \pi + \lambda x^a y^b z^c \phi^d\right). \tag{5.1.43}$$

This kind of transformation also appears in the Carrollian limits of both Maxwell and ModMax theories and will be fully explained then.

It must be noted that although Carrollian boosts were not found by employment of this method it is a symmetry of this limit and has generators $\mathcal{K}_A \in \Gamma(T\mathcal{F}_m)$ given by

$$\mathcal{K}_A = x_A \frac{\partial}{\partial s}. \tag{5.1.44}$$

With transformations written in terms of flow

$$h^{\mathcal{K}_A} : \mathbb{R} \times \mathcal{F}_m \longrightarrow \mathcal{F}_m \tag{5.1.45}$$

$$(\lambda, s, \mathbf{x}, \phi, \pi) \longrightarrow h^{\mathcal{K}_A} (\lambda, s, \mathbf{x}, \phi, \pi) = (s + \lambda x_A, \mathbf{x}, \phi, \pi). \tag{5.1.46}$$

Furthermore, the infinite-dimensional extension of this is also a symmetry of this limit with generators $\mathcal{T}_{abc} \in \Gamma(T\mathcal{F}_m)$ given by

$$\mathcal{T}_{abc} = x^a y^b z^c \frac{\partial}{\partial s}. \tag{5.1.47}$$

Where $a, b, c \in \mathbb{N}_0$ are natural numbers. Symmetry transformations of these vector fields are given by

$$h^{\mathcal{T}_{abc}} : \mathbb{R} \times \mathcal{F}_m \longrightarrow \mathcal{F}_m \tag{5.1.48}$$

$$(\lambda, s, \mathbf{x}, \phi, \pi) \longrightarrow h^{\mathcal{T}_{abc}} (\lambda, s, \mathbf{x}, \phi, \pi) = \left(s + \lambda x^a y^b z^c, \mathbf{x}, \phi, \pi\right). \tag{5.1.49}$$

5.1.2 Electric limit of the scalar free field

As was previously stated, for arriving at the electric Carrollian limit of free scalar theory it is needed to first re-scale the canonically conjugate pair (ϕ, π)

$$\tilde{\pi} = c\pi \qquad \tilde{\phi} = \frac{1}{c}\phi. \tag{5.1.50}$$

In so doing, we transform the action principle as follows

$$S\left[\tilde{\pi}, \tilde{\phi}\right] = \int_{\mathbb{R}^4} d^4x \left[\frac{1}{2}\tilde{\pi}^2 - \frac{c^2}{2}\nabla\tilde{\phi} \cdot \nabla\tilde{\phi}\right], \tag{5.1.51}$$

so of course,

$$S_E[\tilde{\pi}, \tilde{\phi}] = \lim_{c \to 0} S\left[\tilde{\pi}, \tilde{\phi}\right] = \int_{\mathbb{R}^4} d^4x \left[\frac{1}{2}\tilde{\pi}^2\right]. \tag{5.1.52}$$

The equations of motion in this case are $\pi = \dot{\phi}$ and $\dot{\pi} = 0$, which reproduce the correct limit.

5.2 Electric and magnetic limits of a scalar field with an analytic potential

If we include an analytic potential $V(\phi) = \sum_{j \in J} a_m \phi^m$ it's clear that under redefinition of the field we must compensate the appearance of powers of c in the series for the electric limit so we postulate

$$a'_m = c^m {a_m}^m, \tag{5.2.1}$$

where ${a_m}^m$ is the previous m-th coupling constant to the m-th power, c is the speed of light and a'_m is the new m-th coupling constant.

This way we can construct the electric and magnetic limit of a self-interacting scalar field as follows

$$S_M[\pi, \phi] = -\int_{\mathbb{R}^4} d^4x \left[\frac{1}{2}\nabla\phi \cdot \nabla\phi + V(\phi)\right] \tag{5.2.2}$$

$$S_E[\pi, \phi] = \int_{\mathbb{R}^4} d^4x \left[\frac{1}{2}\pi^2 - V(\phi)\right]. \tag{5.2.3}$$

Part II

Maxwell theory, symmetries and limits

Chapter 6

Maxwell theory

The way we usually think about Maxwell's equations is in their standard vector calculus form. We see them and remember our dear Griffiths Griffiths (2017), it's burnt into our retinas.

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0 \tag{6.0.1}$$

$$\nabla \cdot \boldsymbol{E} = \rho/\varepsilon_0 \qquad \nabla \times \boldsymbol{B} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = \mu_0 \boldsymbol{J}, \tag{6.0.2}$$

where $\boldsymbol{B}$, $\boldsymbol{E}$ and $\boldsymbol{J}$ are $\mathbb{R}^3$ valued vector fields[1] and ρ is a real valued function. In this work we are mainly concerned about vacuum Maxwell equations, which have no sources. This is $\rho = 0$ and $\boldsymbol{J} = 0$, therefore

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0 \tag{6.0.3}$$

$$\nabla \cdot \boldsymbol{E} = 0 \qquad \nabla \times \boldsymbol{B} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = 0. \tag{6.0.4}$$

6.1 Symmetries

Maxwell theory has been studied from numerous approaches and one particularly important is that of its symmetries. Although not immediately clear from the

[1] In the vector calculus sense.

equations of motion there are three symmetries involved in them:

- Poincaré symmetry both in the vacuum case and in the one with sources.
- Conformal symmetry in the vacuum case.
- Duality invariance in the vacuum case.

In what follows, the Lie point symmetry approach was used to compute these symmetries by first solving an overdetermined system of PDEs to find those vector fields that generate said symmetries and then using them to construct the flows associated to them by solving for the integral curves[2]. Symmetries are grouped according to natural subgroups of the total symmetry group.

We consider the fiber bundle $(\mathcal{E}, \pi, M)$, where M is the four-dimensional Minkowski space-time and π is the projection map

$$\pi : \mathcal{E} \longrightarrow M \tag{6.1.1}$$

$$(t, x, y, z, E_1, E_2, E_3, B_1, B_2, B_3) \longrightarrow (t, x, y, z)\,. \tag{6.1.2}$$

For simplicity the notation $\mathbf{x} = (x, y, z)$, $\boldsymbol{E} = (E_1, E_2, E_3)$ and $\boldsymbol{B} = (B_1, B_2, B_3)$ will also be used in this work, an example of it would be writing (6.1.1) as

$$\pi : \mathcal{E} \longrightarrow M \tag{6.1.3}$$

$$(t, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) \longrightarrow (t, \mathbf{x})\,, \tag{6.1.4}$$

which helps clean up the notation. Equations of motion define a region $O \subseteq \mathcal{E}$ for which symmetry transformations are endomorphisms.

6.1.1 Lorentz

Lorentzian symmetry consists on spatial translations, time translations, space rotations and boosts as described in section 3.2. In the following, the action of

[2]By following this approach only the connected part to the identity can be recovered.

this group on the electric and magnetic field is constructed. Spatial translations have generators $\mathcal{P}_A \in T\mathcal{E}$ given by

$$\mathcal{P}_A = \frac{\partial}{\partial x^A}. \tag{6.1.5}$$

Let $p \in \mathcal{E}$ be a point $p = (t_0, x_0, y_0, z_0, E_1^0, E_2^0, E_3^0, B_1^0, B_2^0, B_3^0)$ to be used as initial conditions for the curve $\gamma_p : \mathbb{R} \longrightarrow \mathcal{E}$ such that $\gamma_p(0) = p$. Solving the system of ODEs

$$\dot{\gamma}_p^{\mathcal{P}_A}(\lambda) = \mathcal{P}_{A\,\gamma_p^{\mathcal{P}_A}(\lambda)}, \tag{6.1.6}$$

we find as the solution the unique curve γ_p that passes through the point p and has tangent vector $\mathcal{P}_{A\,\gamma_p}$. We use this curve to construct the flows

$$h^{\mathcal{P}_A} : \mathbb{R} \times \mathcal{E} \longrightarrow \mathcal{E} \tag{6.1.7}$$
$$(\lambda, e) \longrightarrow h^{\mathcal{P}_A}(\lambda, e) := \gamma_e^{\mathcal{P}_A}(\lambda). \tag{6.1.8}$$

By doing so, the flow for each spatial translation is constructed as listed below

$$h^{\mathcal{P}_1}(\lambda, t, x, y, z, E_1, E_2, E_3, B_1, B_2, B_3) = (t, x + \lambda, y, z, E_1, E_2, E_3, B_1, B_2, B_3) \tag{6.1.9}$$
$$h^{\mathcal{P}_2}(\lambda, t, x, y, z, E_1, E_2, E_3, B_1, B_2, B_3) = (t, x, y + \lambda, z, E_1, E_2, E_3, B_1, B_2, B_3) \tag{6.1.10}$$
$$h^{\mathcal{P}_3}(\lambda, t, x, y, z, E_1, E_2, E_3, B_1, B_2, B_3) = (t, x, y, z + \lambda, E_1, E_2, E_3, B_1, B_2, B_3). \tag{6.1.11}$$

Time translations behave in much the same way as their spatial counterpart[3]. The vector field generating this transformation is $\mathcal{H} \in T\mathcal{E}$

[3]Meaning they have no effect on the fields and they also act as an additive $\mathbb{R}$-group.

$$\mathcal{H} = \frac{1}{c}\frac{\partial}{\partial t}. \tag{6.1.12}$$

The system of ordinary differential equations to solve in this case is

$$\dot{\gamma}_p^{\mathcal{H}}(\lambda) = \mathcal{H}_{\gamma_p^{\mathcal{H}}(\lambda)}. \tag{6.1.13}$$

This has a unique solution for initial conditions $\gamma_p^{\mathcal{H}}(0) = p$ which are used to construct the flow as

$$h^{\mathcal{H}} : \mathbb{R} \times \mathcal{E} \longrightarrow \mathcal{E} \tag{6.1.14}$$

$$(\lambda, e) \longrightarrow h^{\mathcal{H}}(\lambda, e) := \gamma_e^{\mathcal{H}}(\lambda). \tag{6.1.15}$$

Explicitly we have

$$h^{\mathcal{H}}(\lambda, t, x, y, z, E_1, E_2, E_3, B_1, B_2, B_3) = (t + \lambda/c, x, y, z, E_1, E_2, E_3, B_1, B_2, B_3). \tag{6.1.16}$$

Rotations are also a symmetry of Maxwell equations, as is implied by stating they are Lorentz invariant. They have generators[4] $\mathcal{J}_A \in T\mathcal{E}$ given by

$$\mathcal{J}_A = \epsilon_{ABC}\left(x^B\frac{\partial}{\partial x^C} + E^B\frac{\partial}{\partial E^C} + B^B\frac{\partial}{\partial B^C}\right). \tag{6.1.17}$$

The system of ordinary differential equations to solve for finding how the finite transformation associated with each $\mathcal{J}_A$ acts is

[4]Although the presentation of these symmetries has been shown from the lens of solving differential equations, it is important to remember to infer how they act based on how the vector looks. In this case it can be concluded both space and fields get rotated in the same way.

$$\dot{\gamma}_p^{\mathcal{J}_A}(\lambda) = \mathcal{J}_{A\,\gamma_p^{\mathcal{J}_A}(\lambda)}. \tag{6.1.18}$$

This has a unique solution with initial conditions $\gamma_p^{\mathcal{J}_A}(0) = p$, which is used to construct the flows

$$h^{\mathcal{J}_A} : \mathbb{R} \times \mathcal{E} \longrightarrow \mathcal{E} \tag{6.1.19}$$

$$(\lambda, e) \longrightarrow h^{\mathcal{J}_A}(\lambda, e) := \gamma_e^{\mathcal{J}_A}(\lambda). \tag{6.1.20}$$

For the first angular momentum generator we have the action of a rotation with respect to the x-axis

$$\begin{aligned} h^{\mathcal{J}_1}(\lambda, t, x, y, z, E_1, E_2, E_3, B_1, B_2, B_3) = \Big(&t, x, y\cos\lambda + z\sin\lambda, z\cos\lambda - y\sin\lambda,\\ &E_1, E_2\cos\lambda + E_3\sin\lambda, E_3\cos\lambda - E_2\sin\lambda,\\ &B_1, B_2\cos\lambda + B_3\sin\lambda, B_3\cos\lambda - B_2\sin\lambda\Big). \end{aligned} \tag{6.1.21}$$

For the second angular momentum generator we have the action of a rotation with respect to the y-axis

$$\begin{aligned} h^{\mathcal{J}_2}(\lambda, t, x, y, z, E_1, E_2, E_3, B_1, B_2, B_3) = \Big(&t, x\cos\lambda - z\sin\lambda, y, z\cos\lambda + x\sin\lambda,\\ &E_1\cos\lambda - E_3\sin\lambda, E_2, E_3\cos\lambda + E_1\sin\lambda,\\ &B_1\cos\lambda - B_3\sin\lambda, B_2, B_3\cos\lambda + B_1\sin\lambda\Big). \end{aligned} \tag{6.1.22}$$

For the third angular momentum generator we have the action of a rotation with respect to the z-axis

$$h^{\mathcal{J}_3}(\lambda, t, x, y, z, E_1, E_2, E_3, B_1, B_2, B_3) = \Big(t, x\cos\lambda + y\sin\lambda, y\cos\lambda - x\sin\lambda, z,$$
$$E_1\cos\lambda + E_2\sin\lambda, E_2\cos\lambda - E_1\sin\lambda, E_3,$$
$$B_1\cos\lambda + B_2\sin\lambda, B_2\cos\lambda - B_1\sin\lambda, B_3\Big). \tag{6.1.23}$$

The remaining part of the Lorentz group are boosts $\mathcal{K}_A \in T\mathcal{E}$ that generate Lorentz transformations in the total space $\mathcal{E}$. These generators are given by

$$\mathcal{K}_A = ct\frac{\partial}{\partial x^A} + \frac{x_A}{c}\frac{\partial}{\partial t} + \epsilon_{ABC}\left(cB^B\frac{\partial}{\partial E^C} - \frac{E^B}{c}\frac{\partial}{\partial B^C}\right). \tag{6.1.24}$$

The system of ODEs to solve in order to build the transformations that come from the generators of boosts is

$$\dot{\gamma}_p^{\mathcal{K}_A}(\lambda) = \mathcal{K}_{A\,\gamma_p^{\mathcal{K}_A}(\lambda)}. \tag{6.1.25}$$

The unique solution of this system of equations with initial conditions $\gamma_p^{\mathcal{K}_A}(0) = p$ is used to construct the flows that serve as the action of boosts

$$h^{\mathcal{K}_A} : \mathbb{R} \times \mathcal{E} \longrightarrow \mathcal{E} \tag{6.1.26}$$
$$(\lambda, e) \longrightarrow h^{\mathcal{K}_A}(\lambda, e) := \gamma_e^{\mathcal{K}_A}(\lambda). \tag{6.1.27}$$

Just as it was seen in a previous chapter, boosts act as hyperbolic rotations on space-time. They act on the fields in much the same way. The flow of the first boost is given by

$$h^{\mathcal{K}_1}(\lambda,t,x,y,z,E_1,E_2,E_3,B_1,B_2,B_3) = \Big(t\cosh\lambda + \frac{x\sinh\lambda}{c}, x\cosh\lambda + ct\sinh\lambda, y, z,$$
$$E_1, E_2\cosh\lambda + cB_3\sinh\lambda, E_3\cosh\lambda - cB_3\sinh\lambda,$$
$$B_1, B_2\cosh\lambda - \frac{E_3\sinh\lambda}{c}, B_3\cosh\lambda + \frac{E_2\sinh\lambda}{c}\Big), \tag{6.1.28}$$

the flow of the second boost is given by

$$h^{\mathcal{K}_2}(\lambda,t,x,y,z,E_1,E_2,E_3,B_1,B_2,B_3) = \Big(t\cosh\lambda + \frac{y\sinh\lambda}{c}, x, y\cosh\lambda + ct\sinh\lambda, z,$$
$$E_1\cosh\lambda - cB_3\sinh\lambda, E_2, E_3\cosh\lambda + cB_1\sinh\lambda,$$
$$B_1\cosh\lambda + \frac{E_1\sinh\lambda}{c}, B_2, B_3\cosh\lambda + \frac{E_1\sinh\lambda}{c}\Big), \tag{6.1.29}$$

and the flow of the third boost is given by

$$h^{\mathcal{K}_3}(\lambda,t,x,y,z,E_1,E_2,E_3,B_1,B_2,B_3) = \Big(t\cosh\lambda + \frac{z\sinh\lambda}{c}, x, y, z\cosh\lambda + ct\sinh\lambda,$$
$$E_1\cosh\lambda + cB_2\sinh\lambda, E_2\cosh\lambda - cB_1\sinh\lambda, E_3,$$
$$B_1\cosh\lambda - \frac{E_2\sinh\lambda}{c}, B_2\cosh\lambda + \frac{E_1\sinh\lambda}{c}, B_3\Big). \tag{6.1.30}$$

As stated previously, each transformation corresponds to a 1-parameter subgroup with parameter $\lambda \in \mathbb{R}$. The action of these transformations can be characterized by defining endomorphisms

$$h^{\mathcal{X}}_{\lambda} : \mathcal{E} \longrightarrow \mathcal{E} \tag{6.1.31}$$
$$e \longrightarrow h^{\mathcal{X}}(\lambda, e). \tag{6.1.32}$$

For any generator $\mathcal{X}$ and real number λ. Composition of such functions forms a group and, in this particular case, an action of the Poincaré group ISO $(3,1)$.

6.1.1.1 Space-time restriction

A representation of the Poincaré group ISO $(3,1)$ is recovered by taking the projection of all endomorphisms defined in the previous section, with

$$h_\lambda^{P_A} = \pi \circ h_\lambda^{\mathcal{P}_A} \tag{6.1.33}$$
$$h_\lambda^{H} = \pi \circ h_\lambda^{\mathcal{H}} \tag{6.1.34}$$
$$h_\lambda^{J_A} = \pi \circ h_\lambda^{\mathcal{J}_A} \tag{6.1.35}$$
$$h_\lambda^{K_A} = \pi \circ h_\lambda^{\mathcal{K}_A}, \tag{6.1.36}$$

where the generators of space-time symmetries can be obtained from the pushforward of the projection map

$$P_A = \pi_* \mathcal{P}_A = \frac{\partial}{\partial x^A} \tag{6.1.37}$$
$$H = \pi_* \mathcal{H} = \frac{1}{c}\frac{\partial}{\partial t} \tag{6.1.38}$$
$$J_A = \pi_* \mathcal{J}_A = \epsilon_{ABC} x^A \frac{\partial}{\partial x^C} \tag{6.1.39}$$
$$K_A = \pi_* \mathcal{K}_A = ct \frac{\partial}{\partial x^A} + \frac{x_A}{c}\frac{\partial}{\partial t}. \tag{6.1.40}$$

This procedure will be used to take the space-time restriction of symmetries found in Carrollian limits.

6.1.2 Conformal

Space-time dilations are also a symmetry of Maxwell equations without sources, much in the same way as in the wave equation case. Space-time dilations have generator $\mathcal{D} \in T\mathcal{E}$ given by

$$\mathcal{D} = x^A \frac{\partial}{\partial x^A} + t\frac{\partial}{\partial t}. \tag{6.1.41}$$

The system of ODEs we need to solve to construct how space-time dilations act is

$$\dot{\gamma}_p^{\mathcal{D}}(\lambda) = \mathcal{D}_{\gamma_p^{\mathcal{D}}(\lambda)}. \tag{6.1.42}$$

This has unique solution with initial conditions $\gamma_p^{\mathcal{D}}(0) = p$ which is used to construct the flow

$$h^{\mathcal{D}} : \mathbb{R} \times \mathcal{E} \longrightarrow \mathcal{E} \tag{6.1.43}$$

$$(\lambda, e) \longrightarrow h^{\mathcal{D}}(\lambda, e) := \gamma_e^{\mathcal{D}}(\lambda). \tag{6.1.44}$$

Explicitly, space-time dilations are given by[5]

$$h^{\mathcal{D}}(\lambda, t, x, y, z, E_1, E_2, E_3, B_1, B_2, B_3) = \left(e^{\lambda}t, e^{\lambda}x, e^{\lambda}y, e^{\lambda}z, E_1, E_2, E_3, B_1, B_2, B_3\right) \tag{6.1.45}$$

Field dilations are also a symmetry of Maxwell's equations, with generator $\mathcal{W} \in T\mathcal{E}$ given by

$$\mathcal{W} = E^A \frac{\partial}{\partial E^A} + B^A \frac{\partial}{\partial B^A}. \tag{6.1.46}$$

[5]This is a point where it would be useful to remind you, dear reader, we are only recovering the connected part to the identity.

The system of ODEs we need to solve to construct how space-time dilations act is

$$\dot{\gamma}_p(\lambda) = \mathcal{W}_{\gamma_p(\lambda)}. \tag{6.1.47}$$

This has unique solution with initial conditions $\gamma_p^{\mathcal{W}}(0) = p$ which is used to construct the flow

$$h^{\mathcal{W}}(\lambda, t, x, y, z, E_1, E_2, E_3, B_1, B_2, B_3) = \left(t, x, y, z, e^{\lambda}E_1, e^{\lambda}E_2, e^{\lambda}E_3, e^{\lambda}B_1, e^{\lambda}B_2, e^{\lambda}B_3\right). \tag{6.1.48}$$

Special conformal transformations were also found to be a symmetry of Maxwell's equations but are not presented here because their explicit form is quite complicated and showing them would not serve advance any understanding on the subject.

6.1.3 Duality

All symmetries so far have involved space-time. However, there is one that do not involve them. This one being duality invariance with generator $\mathcal{U} \in T\mathcal{E}$ given by

$$\mathcal{U} = -cB^A \frac{\partial}{\partial E^A} + \frac{E^A}{c}\frac{\partial}{\partial B^A}. \tag{6.1.49}$$

The system of ordinary differential equations to solve is

$$\dot{\gamma}_p^{\mathcal{U}}(\lambda) = \mathcal{U}_{\gamma_p^{\mathcal{U}}(\lambda)}. \tag{6.1.50}$$

This has unique solution with initial conditions $\gamma_p^{\mathcal{U}}(0) = p$ which is used to construct the appropriate transformation via flow

$$h^{\mathcal{U}} : \mathbb{R} \times \mathcal{E} \longrightarrow \mathcal{E} \tag{6.1.51}$$

$$(\lambda, e) \longrightarrow h^{\mathcal{U}}(\lambda, e) := \gamma_e^{\mathcal{U}}(\lambda) . \tag{6.1.52}$$

Explicitly, duality transformations act as the action of the rotation group in the $(\boldsymbol{E}, \boldsymbol{B})$ pair

$$h^{\mathcal{U}}(\lambda, t, x, y, z, \boldsymbol{E}, \boldsymbol{B}) = \left(t, x, y, z, \boldsymbol{E}\cos\lambda - c\boldsymbol{B}\sin\lambda, \boldsymbol{B}\cos\lambda + \frac{1}{c}\boldsymbol{E}\sin\lambda\right). \tag{6.1.53}$$

6.2 Lagrangian formulation

Vacuum Maxwell equations come from two different places. Half of them are a consequence of considering electrodynamics as a gauge theory of the group $U(1)$ with curvature F[6]. The other half are the equations of motion derived from the action principle over a region of Lorentzian space-time $\Omega \subseteq M$

$$S[A, dA] = \frac{1}{2}\int_\Omega F \wedge \star F = -\frac{1}{2}\int_\Omega \langle F, F\rangle \,\omega_g. \tag{6.2.1}$$

6.2.1 First pair of equations: the Bianchi identity

The tensor $F = dA$ is a real valued $U(1)$ curvature for the connection 1-form A, with

[6]Since $U(1)$ has no group index, there's no need to take the trace of $F \wedge \star F$.

$$
\begin{aligned}
A =& -\phi dt + A_x dx + A_y dy + A_z dz \qquad (6.2.2)\\
F =& \left(\frac{\partial A_x}{\partial t} + \frac{\partial \phi}{\partial x}\right) dt \wedge dx + \left(\frac{\partial A_y}{\partial t} + \frac{\partial \phi}{\partial y}\right) dt \wedge dy + \left(\frac{\partial A_z}{\partial t} + \frac{\partial \phi}{\partial z}\right) dt \wedge dz \qquad (6.2.3)\\
&+ \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) dy \wedge dz + \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) dx \wedge dz + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) dx \wedge dy\\
=& -E_x dt \wedge dx - E_y dt \wedge dy - E_z dt \wedge dz + B_x\, dy \wedge dz - B_y\, dx \wedge dz + B_z\, dx \wedge dy \qquad (6.2.4)\\
=& E \wedge dt + B. \qquad (6.2.5)
\end{aligned}
$$

Now, this has matrix elements[7]

$$
F_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}. \qquad (6.2.6)
$$

For a $U(1)$ theory, the Bianchi identity is expressed as

$$
\begin{aligned}
d_A F =& dF = dd_A A = ddA = 0 \qquad (6.2.7)\\
=& \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) dt \wedge dx \wedge dy - \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) dt \wedge dx \wedge dz\\
&+ \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) dt \wedge dy \wedge dz\\
&+ \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) dx \wedge dy \wedge dz + \frac{\partial B_x}{\partial t} dt \wedge dy \wedge dz\\
&- \frac{\partial B_y}{\partial t} dt \wedge dx \wedge dz + \frac{\partial B_z}{\partial t} dt \wedge dx \wedge dy. \qquad (6.2.8)
\end{aligned}
$$

[7] Taking $x^0 = ct$.

Using linear independence and regrouping terms we arrive at

$$\nabla \cdot \boldsymbol{B} = 0 \qquad\qquad \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0. \tag{6.2.9}$$

The second pair of Maxwell equations come from an action principle, for which it is convenient to compute the Hodge dual $\bar{F} = \star F$. This is carried over by using our previous definition as

$$\star F = \frac{1}{2}\epsilon_{\mu_1\mu_2\mu_3\mu_4}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}F_{\nu_1\nu_2}dx^{\mu_3} \wedge dx^{\mu_4}, \tag{6.2.10}$$

this has matrix representation given by

$$(\star F)_{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix}. \tag{6.2.11}$$

Note that if we perform two consecutive Hodge star operations we arrive at the same fields but multiplied by minus one.

6.2.2 Second pair of equations: Lagrangian's EOM

While the first pair of Maxwell's equations come from a Bianchi identity for a $U(1)$ theory and must be satisfied if we wish to claim to be working in a gauge theory setting, this is not the case for the second pair, which are derived from a particular Lagrangian. Namely

$$\mathcal{S}[A, F] = \int_\Omega \frac{1}{2}F \wedge *F - A \wedge *J. \tag{6.2.12}$$

Where $\Omega \subseteq M$ is a closed submanifold of Minkowski space-time[8].

This action principle can, of course, be written in quite different ways. One that works quite well for showing Lorentz invariance comes from using the defining property of the Hodge dual Flanders (1963) to rewrite the previous equation. Let ω_g be the volume form associated to the Lorentzian metric g, then

$$\mathcal{S}[A, F] = \int_\Omega -\frac{1}{2} \langle F, F \rangle \, \omega_g + \langle A, J \rangle \, \omega_g, \tag{6.2.13}$$

where $\langle \cdot, \cdot \rangle : \Omega^2(M) \times \Omega^2(M) \to C^\infty(M)$ is a $SO(1,3)$-invariant pseudo inner product. It is clear then that this is, by construction, Lorentz invariant.

We take the connection A to be such that the action takes an extremal value. We then consider a one parameter family of connections

$$A(\alpha) := A + \alpha a. \tag{6.2.14}$$

This way, $A(0) = A$ extremizes the action. We construct the functional derivative in an analogous way as in Goldstein et al. (2002). First, we take the following difference

$$\begin{aligned}
\mathcal{S}[A(\alpha)] - \mathcal{S}[A(0)] &= \int_\Omega -\frac{1}{2} \langle F + \alpha da, F + \alpha da \rangle \, \omega_g + \langle A + \alpha a, J \rangle \, \omega_g \\
&\quad - \int_\Omega -\frac{1}{2} \langle F, F \rangle \, \omega_g + \langle A, J \rangle \, \omega_g \qquad (6.2.15)\\
&= \alpha \int_\Omega - \langle F, da \rangle \, \omega_g + \langle a, J \rangle \, \omega_g - \frac{1}{2}\alpha^2 \int_\Omega \langle da, da \rangle \, \omega_g. \qquad (6.2.16)
\end{aligned}$$

Next, we divide by $\alpha \neq 0$ and take the limit $\alpha \to 0$

[8] This requirement is not strictly necesssary and is often dropped.

$$\begin{aligned}
\frac{d\mathcal{S}[A]}{d\alpha}(0) &:= \lim_{\alpha\to 0}\frac{S[A(\alpha)]-S[A(0)]}{\alpha} && (6.2.17)\\
&= \int_\Omega \left(-\langle F, da\rangle + \langle J, a\rangle\right)\omega_g && (6.2.18)\\
&= \int_\Omega *F\wedge da - *J\wedge a + d*F\wedge a - d*F\wedge a && (6.2.19)\\
&= \int_{\partial\Omega} *F\wedge a - \int_\Omega (d*F + *J)\wedge a. && (6.2.20)
\end{aligned}$$

Now, since we are physicists we are blind to boundary terms[9]. Also, a is an arbitrary 1-form so for the action to have an extremal value we need

$$d*F + *J = 0. \tag{6.2.21}$$

This can also be seen if we write

$$\frac{d\mathcal{S}[A]}{d\alpha}(0) = -\int_\Omega \langle d*F + *J, a\rangle\,\omega_g. \tag{6.2.22}$$

The bilinear form $\langle\cdot,\cdot\rangle$ is non-degenerate, so $d*F+*J$ must be 0. Notice here the way we arrived at the equations of motion is unique, the reason for that is the explicit presence of A in the Lagrangian. If it weren't for it we could have carried this procedure in two different ways, as I'll show in the section on duality.

6.3 Hamiltonian formulation

Hamiltonian formulations of gauge theories must be done carefully because they are constrained systems. Bianchi's identity

[9]This is not exactly true. There are plenty of instances where boundary terms are relevant, specially when considering physics of materials, which in electrodynamics is quite a relevant area since it has direct impact on how we transmit signals.

$$d_A F = 0 \tag{6.3.1}$$

implies that not all of configuration space is accessible and, therefore, not all of phase space is accessible and we must deal with this. Standard procedure is Dirac's, who presented a systematic way of extending the Hamiltonian with Lagrange multipliers in Dirac (2001) so that the equations of motion obtained from the extended Hamiltonian coincide with those obtained from the Lagrangian formulation.

Maxwell's Hamiltonian is constructed as usual. First we use it's Lagrangian to build the canonical momenta

$$\pi^a = \frac{\partial \mathcal{L}}{\partial \dot{A}_a} = -\frac{1}{c^2} E^a. \tag{6.3.2}$$

Since $\boldsymbol{E} = -\dot{\boldsymbol{A}} - \nabla\phi$, we can solve for $\dot{\boldsymbol{A}}$ as follows

$$\dot{\boldsymbol{A}} = c^2 \pi - \nabla\phi. \tag{6.3.3}$$

Notice there's no appearance of $\dot{\phi}$ in the Lagrangian, so we have the restriction $\pi^0 = 0$. In Dirac's jargon this equation is referred to as a primary constraint. We will come back to this equation later.

Now the Lagrangian must be expressed in terms of canonical variables. To do this it is quite convenient to separate the Lagrangian density as follows

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \tag{6.3.4}$$

$$= \frac{1}{2}\left(\frac{E^2}{c^2} - B^2\right) \tag{6.3.5}$$

$$= \frac{1}{2} c^2 \pi^a \pi_a - \frac{1}{4} F^{ab} F_{ab}. \tag{6.3.6}$$

With this, we get the following Hamiltonian

$$H = \int_\Omega \left[\pi \cdot \boldsymbol{A} - \mathcal{L}\right] d^3\mathbf{x} \tag{6.3.7}$$

$$H = \int_\Omega \left[\frac{1}{2}c^2\pi^a\pi_a + \frac{1}{4}F^{ab}F_{ab} - \pi \cdot \nabla\phi\right] d^3\mathbf{x}. \tag{6.3.8}$$

At this point it is convenient to integrate by parts the last term. The objective of this is twofold, firstly we can see more clearly how the scalar field behaves in the Hamiltonian and secondly it makes it easier to arrive at Gauss law. This yields

$$H = \int_\Omega \left[\frac{1}{2}c^2\pi^a\pi_a + \frac{1}{4}F^{ab}F_{ab} + \phi\,\nabla \cdot \pi\right] d^3\mathbf{x}. \tag{6.3.9}$$

The equations of motion for the field ϕ that come from this Hamiltonian are

$$\dot{\phi} = \frac{\partial \mathcal{H}}{\partial \pi^0} - \frac{\partial}{\partial x^j}\left(\frac{\partial \mathcal{H}}{\partial(\partial_j \pi^0)}\right) = 0 \tag{6.3.10}$$

and

$$\dot{\pi}^0 = -\frac{\partial \mathcal{H}}{\partial \phi} + \frac{\partial}{\partial x^j}\left(\frac{\partial \mathcal{H}}{\partial(\partial_j \phi)}\right) \tag{6.3.11}$$

$$= -\nabla \cdot \pi. \tag{6.3.12}$$

Taking (6.3.10) and (6.3.12) we arrive at the usual Gauss equation

$$-\nabla \cdot \boldsymbol{\pi} = 0 \tag{6.3.13}$$

$$\frac{1}{c^2}\nabla \cdot \boldsymbol{E} = 0. \tag{6.3.14}$$

But equation (6.3.10) can't be right since it implies there's no time variation for the scalar potential. This inconsistency comes, as has been pointed out, from the fact that variations are restricted by Bianchi's identity.

The equations of motion for the vector potential are obtained next

$$\dot{A}_a = \frac{\partial \mathcal{H}}{\partial \pi^a} - \frac{\partial}{\partial x^j}\left(\frac{\partial \mathcal{H}}{\partial(\partial_j \pi^a)}\right) \tag{6.3.15}$$

$$= c^2 \pi_a - \partial_a \phi, \tag{6.3.16}$$

this is just a restatement of the definition of the canonical momentum π^a in (6.3.2). The last equation is

$$\dot{\pi}^a = -\frac{\partial \mathcal{H}}{\partial A_a} + \frac{\partial}{\partial x^i}\left(\frac{\partial \mathcal{H}}{\partial\left(\partial_i A_a\right)}\right) \tag{6.3.17}$$

$$= \frac{\partial}{\partial x^i}\left(B_k \frac{\partial}{\partial\left(\partial_i A_a\right)} \epsilon^{lmk} \partial_l A_m\right) \tag{6.3.18}$$

$$-\frac{1}{c^2}\frac{\partial \boldsymbol{E}^a}{\partial t} = -\left(\nabla \times \boldsymbol{B}\right)^a. \tag{6.3.19}$$

Rearranging terms we obtain the Ampere-Maxwell equation

$$\nabla \times \boldsymbol{B} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = 0. \tag{6.3.20}$$

Equations (6.3.16) and (6.3.20) together with those from Bianchi's identity form the four equations of Maxwell's electrodynamics. However, we must deal with the inconsistency we encountered. As has been pointed out a number of times, we need to use Dirac's formalism. In Dirac's jargon, equation $\pi^0 = 0$ is a primary constraint and equation (6.3.16) is a secondary constraint[10].

To include these constraints in the formulation, the use of Lagrange multipliers is needed. We define the extended Hamiltonian H^{ex} as

[10] Known as the Gauss constraint. The requirement that this constraint is preserved in time gives rise to the conservation of electric charge.

$$
\begin{aligned}
H^{ex} :&= H + \int_\Omega \left[\mathghost\, \nabla \cdot \boldsymbol{\pi} + \mathpumpkin\, \pi^0\right] d^3\mathbf{x} && (6.3.21)\\
&= \int_\Omega \left[\frac{1}{2}c^2\pi^a\pi_a + \frac{1}{4}F^{ab}F_{ab} + (\phi + \mathghost)\ \nabla \cdot \pi + \mathpumpkin\, \pi^0\right] d^3\mathbf{x}. && (6.3.22)
\end{aligned}
$$

Where $\mathpumpkin$ and $\mathghost$ are Lagrangian multipliers. As Zangwill shows in Zangwill (2013), this Hamiltonian reproduces the appropriate equations of motion and it can be proved that these Lagrange multipliers are responsible of Gauge-fixing.

Chapter 7

Carrollian limits

7.1 At the level of the equations of motion

The first appearance of Carrollian limits in the literature were taken directly from the equations of motion by appropriate previous redefinition of the fields so said limits exist[1]. We shall reproduce this approach here and answer the question

What are all the symmetries of these limits?

So far, all theories have two distinct limits called magnetic and electric referring to the electromagnetic case. We shall start exploring the Carrollian magnetic limit of Maxwell's equations.

7.1.1 Magnetic limit

We consider vacuum Maxwell's equations written in a slightly different fashion which is useful for taking the Carrollian limits

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0 \qquad \nabla \cdot \boldsymbol{E} = 0 \tag{7.1.1}$$

$$-c^2 \nabla \times \boldsymbol{B} + \frac{\partial \boldsymbol{E}}{\partial t} = 0 \qquad \nabla \cdot \boldsymbol{B} = 0. \tag{7.1.2}$$

From this, it is possible to arrive at the correct limit by simply considering

[1]See Duval et al. (2014c), for example.

$c \to 0$. However, let us use Carrollian field and time reparametrizations given by $\boldsymbol{B} = (C/c)\, \boldsymbol{B_m}$, $\boldsymbol{E} = \boldsymbol{E_m}$ and $s = (cC)\, t$, respectively.

$$\nabla \times \boldsymbol{E_m} + C^2 \frac{\partial \boldsymbol{B_m}}{\partial s} = 0 \qquad \nabla \cdot \boldsymbol{E_m} = 0 \tag{7.1.3}$$

$$(cC)\left(-\nabla \times \boldsymbol{B} + \frac{\partial \boldsymbol{E_m}}{\partial s}\right) = 0 \qquad \nabla \cdot \boldsymbol{B_m} = 0. \tag{7.1.4}$$

The Ampere-Maxwell equation is multiplied by a $(cC)^{-1}$ factor and the Faraday equation by a C^{-2} factor. Then the limit $C \to \infty$ is taken, arriving at the Carrollian magnetic limit of Maxwell's equations

$$\nabla \times \boldsymbol{B_m} - \frac{\partial \boldsymbol{E_m}}{\partial s} = 0 \qquad \nabla \cdot \boldsymbol{E_m} = 0 \tag{7.1.5}$$

$$\frac{\partial \boldsymbol{B_m}}{\partial s} = 0 \qquad \nabla \cdot \boldsymbol{B_m} = 0. \tag{7.1.6}$$

These equations are known to be invariant under the flat Carrollian group, consisting of time translations, spatial translations, spatial rotations and Carrollian boosts.

Neither time translations nor space translations come with field transformations. Rotations act in the same way as in every other vector field, Carrollian boosts in the magnetic limit act as

$$\boldsymbol{B_m}(\boldsymbol{x}, s) \to \boldsymbol{B'_m}(\boldsymbol{x}, s) = \boldsymbol{B_m}(\boldsymbol{x}, s - \boldsymbol{b} \cdot \boldsymbol{x}) \tag{7.1.7}$$

$$\boldsymbol{E_m}(\boldsymbol{x}, s) \to \boldsymbol{E'_m}(\boldsymbol{x}, s) = \boldsymbol{E_m}(\boldsymbol{x}, s - \boldsymbol{b} \cdot \boldsymbol{x}) + \boldsymbol{b} \times \boldsymbol{B_m}(\boldsymbol{x}, s - \boldsymbol{b} \cdot \boldsymbol{x}). \tag{7.1.8}$$

These are, however, not all symmetry transformations of this system of differential equations.

7.1.1.1 Symmetries of the magnetic limit

The approach taken to obtain the symmetries of this system of differential equations is that of Lie point symmetries, which requires thinking of differential equations as conditions taking place in a space which contains both independent variables $(s, \mathbf{x})$ and dependent variables $(\boldsymbol{E}, \boldsymbol{B})$[2]. Let $(\mathcal{E}_m, \pi_m, C^{3+1})$ be the fiber bundle with base space C^{3+1} and projection map

$$\pi_m : \mathcal{E}_m \longrightarrow C^{3+1} \tag{7.1.9}$$

$$(s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) \longrightarrow \pi_m \left(s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}\right) := \left(s, \mathbf{x}\right), \tag{7.1.10}$$

whose tangent bundle $T\mathcal{E}_m$ is used to construct the equations of motion. The EOM define a region of the tangent bundle that is formed by solutions of the system. Symmetry transformations are endomorphisms on $T\mathcal{E}_m$ that also are endomorphisms on these regions.

The Lie point symmetry method, as described in Cantwell (2002) and summarized and exemplified in appendix A, was used to generate the set of partial differential equations which has as solutions the vector coefficients that generates the symmetries of the system. These solutions were found by employment of polynomial expansions given said system is highly overdetermined. The families of vectors are identified as follows.

The Carrollian magnetic limit of Maxwell theory is invariant under spatial translations, with generators $\mathcal{P}_A$ valued in the tangent space $T\mathcal{E}_m$

$$\mathcal{P}_A = \frac{\partial}{\partial x^A}. \tag{7.1.11}$$

Each of them generates a one-parameter subgroup of transformations that can be found by constructing the flows associated to each $\mathcal{P}_A$. Let γ be a curve on the total space $\mathcal{E}_m$

[2]Where the m subscript has been dropped for simplicity when displaying formulae. This should not be a cause of confusion since all maps and space are adequately labeled.

$$\gamma : \mathbb{R} \longrightarrow \mathcal{E}_m \tag{7.1.12}$$

$$\lambda \longrightarrow \gamma(\lambda). \tag{7.1.13}$$

For γ to be an integral curve of $\mathcal{P}_A$, the following system of ordinary differential equations must be satisfied

$$\dot{\gamma}(\lambda) = \mathcal{P}_{A\,\gamma(\lambda)}. \tag{7.1.14}$$

For each value of A we get one solution that is used to build its corresponding flow by explicit use of the integral curves γ

$$h^{\mathcal{P}_1} : \mathbb{R} \times \mathcal{E}_m \longrightarrow \mathcal{E}_m \tag{7.1.15}$$

$$(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) \longrightarrow h^{\mathcal{P}_1}(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) := (s, x + \lambda, y, z, \boldsymbol{E}, \boldsymbol{B}) \tag{7.1.16}$$

$$h^{\mathcal{P}_2} : \mathbb{R} \times \mathcal{E}_m \longrightarrow \mathcal{E}_m \tag{7.1.17}$$

$$(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) \longrightarrow h^{\mathcal{P}_2}(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) := (s, x, y + \lambda, z, \boldsymbol{E}, \boldsymbol{B}) \tag{7.1.18}$$

$$h^{\mathcal{P}_2} : \mathbb{R} \times \mathcal{E}_m \longrightarrow \mathcal{E}_m \tag{7.1.19}$$

$$(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) \longrightarrow h^{\mathcal{P}_3}(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) := (s, x, y, z + \lambda, \boldsymbol{E}, \boldsymbol{B}). \tag{7.1.20}$$

These transformations can be summarized as follows

$$h^{\mathcal{P}} : \mathbb{R}^3 \times \mathcal{E}_m \longrightarrow \mathcal{E}_m \tag{7.1.21}$$

$$(\boldsymbol{\lambda}, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) \longrightarrow h^{\mathcal{P}}(\boldsymbol{\lambda}, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) := (s, \mathbf{x} + \boldsymbol{\lambda}, \boldsymbol{E}, \boldsymbol{B}). \tag{7.1.22}$$

We see that spatial translations affect neither of the fields. This is consistent with what we already knew from previous works[3].

[3]And from basic field theory. Were we to find field transformations coming from finite translations, we then would have known a mistake had taken place.

It is also customary to write flows as endomorphisms by defining

$$h_\lambda^{\mathcal{X}} : \mathcal{E}_m \longrightarrow \mathcal{E}_m \tag{7.1.23}$$
$$e \longrightarrow h_\lambda^{\mathcal{X}}(e) := h^{\mathcal{X}}(\lambda, e). \tag{7.1.24}$$

Time translations are also a symmetry of this limit, with generator given by

$$\mathcal{H} = \frac{\partial}{\partial s}. \tag{7.1.25}$$

The following system of ODEs is solved to construct the appropriate transformation

$$\dot{\gamma}\left(\lambda\right) = \mathcal{H}_{\gamma(\lambda)}. \tag{7.1.26}$$

Solving this we can use the curve γ to construct the flow

$$h^{\mathcal{H}} : \mathbb{R} \times \mathcal{E}_m \longrightarrow \mathcal{E}_m \tag{7.1.27}$$
$$\left(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}\right) \longrightarrow h^{\mathcal{H}}\left(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}\right) := \left(s + \lambda, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}\right). \tag{7.1.28}$$

Notice in both these cases the transformations do not affect the values of the fields but only the space-time part. That is not the case for rotations. The generators $\mathcal{J}_A$ of rotations in $\mathcal{E}_m$ are

$$\mathcal{J}_A = \epsilon_{ABC}\left(x^B \frac{\partial}{\partial x^C} + E^B \frac{\partial}{\partial E^C} + B^B \frac{\partial}{\partial B^C}\right). \tag{7.1.29}$$

As usual, solving for the integral curves is a necessary step for building the transformations

$$\dot{\gamma}(\lambda) = \mathcal{J}_{A\,\gamma(\lambda)}. \tag{7.1.30}$$

The solution of this system of ODEs with initial values is used to construct the flows. Each of them is a map

$$h^{\mathcal{J}_A} : \mathbb{R} \times \mathcal{E}_m \longrightarrow \mathcal{E}_m \tag{7.1.31}$$

$$(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) \longrightarrow h^{\mathcal{J}_A}(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}), \tag{7.1.32}$$

with

$$\begin{aligned} h^{\mathcal{J}_1}(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) := \Big(& s, x, y\cos\lambda + z\sin\lambda, z\cos\lambda - y\sin\lambda, \\ & E_1, E_2\cos\lambda + E_3\sin\lambda, E_3\cos\lambda - E_2\sin\lambda, \\ & B_1, B_2\cos\lambda + B_3\sin\lambda, B_3\cos\lambda - B_2\sin\lambda\Big), \end{aligned} \tag{7.1.33}$$

where it is possible to identify from this a rotation of angle λ with respect to the x-axis; the second is given by

$$\begin{aligned} h^{\mathcal{J}_2}(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) := \Big(& s, x\cos\lambda - z\sin\lambda, y, z\cos\lambda + x\sin\lambda, \\ & E_1\cos\lambda - E_3\sin\lambda, E_2, E_3\cos\lambda + E_1\sin\lambda, \\ & B_1\cos\lambda - B_3\sin\lambda, B_2, B_3\cos\lambda + B_1\sin\lambda\Big), \end{aligned} \tag{7.1.34}$$

which in this case it corresponds to a rotation of an angle λ with respect to the y-axis; the third one is given by

$$
\begin{aligned}
h^{\mathcal{J}_3}\left(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}\right) := \Big(& s, x\cos\lambda + y\sin\lambda, y\cos\lambda - x\sin\lambda, z, \\
& E_1\cos\lambda + E_2\sin\lambda, E_2\cos\lambda - E_1\sin\lambda, E_3, \\
& B_1\cos\lambda + B_2\sin\lambda, B_2\cos\lambda - B_1\sin\lambda, B_3\Big), \qquad (7.1.35)
\end{aligned}
$$

which, unsurprisingly, is a rotation of angle λ with respect to the z-axis. The three spatial rotations can be summarized in the following function

$$
\begin{aligned}
h^{\mathcal{J}} : SO(3) \times \mathcal{E}_m &\longrightarrow \mathcal{E}_m \qquad (7.1.36)\\
\left(R, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}\right) &\longrightarrow h^{\mathcal{J}}\left(R, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}\right) := \left(s, R\mathbf{x}, R\boldsymbol{E}, R\boldsymbol{B}\right), \qquad (7.1.37)
\end{aligned}
$$

where $SO(3)$ stands there as the connected part to the identity of the orthogonal group in three dimensions. Up until this point we already knew how the symmetries acted on both space-time coordinates and the electric and magnetic field.

In turn, the vector fields $\mathcal{T}_{abc}$, with $a, b, c \in \mathbb{N}_0$ are a generalization of Carrollian boosts. For each value of a, b and c we have

$$
\mathcal{T}_{abc} = x^a y^b z^c \frac{\partial}{\partial s} - \epsilon_{IJK} \frac{\partial x^a y^b z^c}{\partial x^I} B^J \frac{\partial}{\partial E^K}. \qquad (7.1.38)
$$

Solving the system of ODEs that require $\mathcal{T}_{abc}$ to be the tangent vector to a curve γ

$$
\dot{\gamma}\left(\lambda\right) = \mathcal{T}_{abc\,\gamma(\lambda)}. \qquad (7.1.39)
$$

We use the solutions to construct the 1-parameter subgroup of transformations given by the flow

$$h^{\mathcal{T}_{abc}} : \mathbb{R} \times \mathcal{E}_m \longrightarrow \mathcal{E}_m \tag{7.1.40}$$

$$(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) \longrightarrow h^{\mathcal{T}_{abc}} (\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) := \left(s + \lambda x^a y^b z^c, \mathbf{x}, \boldsymbol{E} - \lambda \nabla \left(x^a y^b z^c\right) \times \boldsymbol{B}, \boldsymbol{B}\right). \tag{7.1.41}$$

Computing the successive application of $h^{\mathcal{T}_{abc}}_{\lambda_{abc}}$, where λ_{abc} are the parameters of each $\mathcal{T}_{abc}$, we get a power-series expansion[4]

$$f(x, y, z) = \sum_{a,b,c \in \mathbb{N}_0} \lambda_{abc} x^a y^b z^c, \tag{7.1.42}$$

which means this infinite sector corresponds to an action of the additive group $(C^\infty(\mathbb{R}^3), +)$. It is possible and convenient to summarize this as

$$h^{\mathcal{T}} : C^\infty\left(\mathbb{R}^3\right) \times \mathcal{E}_m \longrightarrow \mathcal{E}_m \tag{7.1.43}$$

$$(f, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) \longrightarrow h^{\mathcal{T}} (f, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) := (s + f(x, y, z), \mathbf{x}, \boldsymbol{E} - \nabla f \times \boldsymbol{B}, \boldsymbol{B}). \tag{7.1.44}$$

This infinite-dimensional sector of the symmetry group corresponds to supertranslations in Carrollian time s and has Carrollian boosts as a subgroup by restricting $f \in C^\infty(\mathbb{R}^3)$ to be a linear function[5].

Space dilations have also be found to be a symmetry of the magnetic Carrollian limit of Maxwell theory, with $T\mathcal{E}_m$-valued generator $\mathcal{D}$ given by

$$\mathcal{D} = x^A \frac{\partial}{\partial x^A} + B^A \frac{\partial}{\partial B^A}. \tag{7.1.45}$$

The system of ODEs to solve is the following

[4]Assuming convergence because why wouldn't we.

[5]Notice with this the function responsible for the symmetry becomes $f(x, y, z) = b_1 x + b_2 y + b_3 z = \boldsymbol{b} \cdot \mathbf{x}$ for $\boldsymbol{b} \in \mathbb{R}^3$ and $\nabla f = \boldsymbol{b}$, which completely recovers the known symmetry transformation of magnetic Carrollian electrodynamics under Carrollian boosts.

$$\dot{\gamma}(\lambda) = \mathcal{D}_{\gamma(\lambda)}. \tag{7.1.46}$$

Using the solution to this, flow $h^{\mathcal{D}}$ is constructed

$$h^{\mathcal{D}} : \mathbb{R} \times \mathcal{E}_m \longrightarrow \mathcal{E}_m \tag{7.1.47}$$

$$(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) \longrightarrow h^{\mathcal{D}}(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) := \left(s, e^{\lambda}\mathbf{x}, \boldsymbol{E}, e^{\lambda}\boldsymbol{B}\right). \tag{7.1.48}$$

In the same fashion, time dilations were found to be a symmetry of this limit. Notice there's a difference in sign in the part responsible for transforming the magnetic field

$$\mathcal{Q} = s\frac{\partial}{\partial s} - B^A \frac{\partial}{\partial B^A}. \tag{7.1.49}$$

You may be wondering, dear reader, how to approach the problem of constructing the associated transformation for time dilations $\mathcal{Q}$. This is quite clearly a problem we have never in our lives tried to solve before. Requiring γ to have tangent vectors given by $\mathcal{Q}$ allows to construct the transformations, this is

$$\dot{\gamma}(\lambda) = \mathcal{Q}_{\gamma(\lambda)}. \tag{7.1.50}$$

Using the solutions to this system of ODEs the flows are built

$$h^{\mathcal{Q}} : \mathbb{R} \times \mathcal{E}_m \longrightarrow \mathcal{E}_m \tag{7.1.51}$$

$$(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) \longrightarrow h^{\mathcal{Q}}(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) := \left(e^{\lambda}s, \mathbf{x}, \boldsymbol{E}, e^{-\lambda}\boldsymbol{B}\right). \tag{7.1.52}$$

Special conformal transformations are also a symmetry of this set of EOMs, with generators $\mathcal{S}_A \in T\mathcal{E}_m$ given by

$$\mathcal{S}_A = 2x_A \left(x^B \frac{\partial}{\partial x^B} + s \frac{\partial}{\partial s} \right) - x_B x^B \frac{\partial}{\partial x^A} \tag{7.1.53}$$

$$- 4x_A E_J \frac{\partial}{\partial E^J} + 2x^J \left(E_A \frac{\partial}{\partial E_J} - E_J \frac{\partial}{\partial E^A} \right) - 2s\, \epsilon_{AJK} B^J \frac{\partial}{\partial E_K} \tag{7.1.54}$$

$$- 4x_A B_J \frac{\partial}{\partial B_J} + 2x^J \left(B_A \frac{\partial}{\partial B_J} - B_J \frac{\partial}{\partial B^A} \right). \tag{7.1.55}$$

Solving the usual set of ODEs

$$\dot{\gamma}\,(\lambda) = \mathcal{S}_{A\,\gamma(\lambda)}, \tag{7.1.56}$$

the flows are constructed. For each value of A we get a 1-parameter subgroup represented by its corresponding flow

$$h^{\mathcal{S}_A} : \mathbb{R} \times \mathcal{E}_m \longrightarrow \mathcal{E}_m \tag{7.1.57}$$

$$(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) \longrightarrow h^{\mathcal{S}_A}\,(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B})\,. \tag{7.1.58}$$

For simplicity in the expressions, let us define

$$\omega_x(\lambda) = \frac{x^2 + y^2 + z^2}{\left(x - \lambda\,(x^2 + y^2 + z^2)\right)^2 + y^2 + z^2} \tag{7.1.59}$$

$$\omega_y(\lambda) = \frac{x^2 + y^2 + z^2}{x^2 + \left(y - \lambda\,(x^2 + y^2 + z^2)\right)^2 + z^2} \tag{7.1.60}$$

$$\omega_z(\lambda) = \frac{x^2 + y^2 + z^2}{x^2 + y^2 + \left(z - \lambda\,(x^2 + y^2 + z^2)\right)^2}. \tag{7.1.61}$$

For convenience, let us also define the accompanying factors

$$\Omega_x(\lambda) = (\lambda x - 1)^2 + \lambda^2\left(y^2+z^2\right) \tag{7.1.62}$$
$$\Omega_y(\lambda) = (\lambda y - 1)^2 + \lambda^2\left(x^2+z^2\right) \tag{7.1.63}$$
$$\Omega_z(\lambda) = (\lambda z - 1)^2 + \lambda^2\left(x^2+y^2\right). \tag{7.1.64}$$

Using these factors we can write two families of matrices that characterize the action of special conformal Carrollian transformations act on the electric and magnetic field. The first family of matrices is $\mathbb{T}_A(\lambda)$, where each one is given by

$$\mathbb{T}_x(\lambda) = \Omega_x(\lambda)\begin{pmatrix} \lambda\left(\lambda x^2 - 2x - \lambda\left(y^2+z^2\right)\right)+1 & 2\lambda y(\lambda x-1) & 2\lambda z(\lambda x - 1) \\ -2\lambda y(\lambda x - 1) & \lambda^2\left(x^2-y^2+z^2\right)-2\lambda x+1 & -2\lambda^2 yz \\ -2\lambda z(\lambda x-1) & -2\lambda^2 yz & \lambda^2\left(x^2+y^2-z^2\right)-2\lambda x+1 \end{pmatrix} \tag{7.1.65}$$

$$\mathbb{T}_y(\lambda) = \Omega_y(\lambda)\begin{pmatrix} \lambda^2\left(-x^2+y^2+z^2\right)-2\lambda y+1 & -2\lambda x(\lambda y-1) & -2\lambda^2 xz \\ 2\lambda x(\lambda y-1) & \lambda\left(-\lambda\left(x^2+z^2\right)+\lambda y^2-2y\right)+1 & 2\lambda z(\lambda y-1) \\ -2\lambda^2 xz & -2\lambda z(\lambda y-1) & \lambda^2\left(x^2+y^2-z^2\right)-2\lambda y+1 \end{pmatrix} \tag{7.1.66}$$

$$\mathbb{T}_z(\lambda) = \Omega_z(\lambda)\begin{pmatrix} \lambda^2\left(-x^2+y^2+z^2\right)-2\lambda z+1 & -2\lambda^2 xy & -2\lambda x(\lambda z-1) \\ -2\lambda^2 xy & \lambda^2\left(x^2-y^2+z^2\right)-2\lambda z+1 & -2\lambda y(\lambda z-1) \\ 2\lambda x(\lambda z-1) & 2\lambda y(\lambda z-1) & \lambda\left(z(\lambda z-2)-\lambda\left(x^2+y^2\right)\right)+1 \end{pmatrix}. \tag{7.1.67}$$

The second family of matrices is $\mathbb{O}_A(\lambda)$, with

$$\mathbb{O}_x(\lambda) = \Omega_x(\lambda)\begin{pmatrix} 0 & 2\lambda^2 sz & -2\lambda^2 sy \\ 2\lambda^2 sz & 0 & -2\lambda s(\lambda x-1) \\ -2\lambda^2 sy & 2\lambda s(\lambda x - 1) & 0 \end{pmatrix} \tag{7.1.68}$$
$$\mathbb{O}_y(\lambda) = \Omega_y(\lambda)\begin{pmatrix} 0 & -2\lambda^2 sz & 2\lambda s(\lambda y-1) \\ -2\lambda^2 sz & 0 & 2\lambda^2 sx \\ -2\lambda s(\lambda y-1) & 2\lambda^2 sx & 0 \end{pmatrix} \tag{7.1.69}$$
$$\mathbb{O}_z(\lambda) = \Omega_z(\lambda)\begin{pmatrix} 0 & -2\lambda s(\lambda z-1) & 2\lambda^2 sy \\ 2\lambda s(\lambda z-1) & 0 & -2\lambda^2 sx \\ 2\lambda^2 sy & -2\lambda^2 sx & 0 \end{pmatrix}. \tag{7.1.70}$$

These matrices have two important properties that will serve in the Carrollian magnetic limit of ModMax theory that are relative to the $\mathbb{R}^3$-inner product

$$(\mathbb{T}_A(\lambda)\boldsymbol{a}) \cdot (\mathbb{T}_A(\lambda)\boldsymbol{b}) = \Omega_A(\lambda)^4 \boldsymbol{a} \cdot \boldsymbol{b} \qquad (\mathbb{T}_A(\lambda)\boldsymbol{a}) \cdot (\mathbb{O}_A(\lambda)\boldsymbol{b}) = 0. \tag{7.1.71}$$

This way, the flow of each special conformal transformation is

$$\begin{aligned} h^{\mathcal{S}_1}(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) &= (\omega_x(\lambda)s, \omega_x(\lambda)(x - \lambda\mathbf{x}\cdot\mathbf{x}), \omega_x(\lambda)y, \omega_x(\lambda)z, \\ &\qquad \mathbb{T}_1(\lambda)\boldsymbol{E} + \mathbb{O}_1(\lambda)\boldsymbol{B}, \mathbb{T}_1(\lambda)\boldsymbol{B}) \qquad (7.1.72) \\ h^{\mathcal{S}_2}(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) &= (\omega_y(\lambda)s, \omega_y(\lambda)x, \omega_y(\lambda)(y - \lambda\mathbf{x}\cdot\mathbf{x}), \omega_y(\lambda)z, \\ &\qquad \mathbb{T}_2(\lambda)\boldsymbol{E} + \mathbb{O}_2(\lambda)\boldsymbol{B}, \mathbb{T}_2(\lambda)\boldsymbol{B}) \qquad (7.1.73) \\ h^{\mathcal{S}_3}(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) &= (\omega_z(\lambda)s, \omega_z(\lambda)x, \omega_z(\lambda)y, \omega_z(\lambda)(z - \lambda\mathbf{x}\cdot\mathbf{x}), \\ &\qquad \mathbb{T}_3(\lambda)\boldsymbol{E} + \mathbb{O}_3(\lambda)\boldsymbol{B}, \mathbb{T}_3(\lambda)\boldsymbol{B}). \qquad (7.1.74) \end{aligned}$$

From now on, the transformations concern solely the electric and magnetic field and leave invariant the space-time part. The first of such transformations is a field dilation with generator given by

$$\mathcal{W} = E^A \frac{\partial}{\partial E^A} + B^A \frac{\partial}{\partial B^A}. \tag{7.1.75}$$

The system of ODEs that determine how the transformation behaves is

$$\dot{\gamma}(\lambda) = \mathcal{W}_{\gamma(\lambda)}. \tag{7.1.76}$$

We use the solutions to this equations to give form to the transformation via flow

$$\begin{aligned} h^{\mathcal{W}} : \mathbb{R} \times \mathcal{E}_m &\longrightarrow \mathcal{E}_m \qquad (7.1.77) \\ (\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) &\longrightarrow h^{\mathcal{W}}(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) := \left(s, \mathbf{x}, e^{\lambda}\boldsymbol{E}, e^{\lambda}\boldsymbol{B}\right). \qquad (7.1.78) \end{aligned}$$

It was seen in Maxwell theory that there was a vector field responsible for duality transformations. Magnetic Carrollian Maxwell theory has an equivalent to that vector field, only it does not produce rotations but rather boosts and it is given by

$$\mathcal{U} = -B^A \frac{\partial}{\partial E^A}. \tag{7.1.79}$$

Solving for γ in

$$\dot{\gamma}(\lambda) = \mathcal{U}_{\gamma(\lambda)}, \tag{7.1.80}$$

we construct the transformation as a flow

$$h^{\mathcal{U}} : \mathbb{R} \times \mathcal{E}_m \longrightarrow \mathcal{E}_m \tag{7.1.81}$$

$$(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) \longrightarrow h^{\mathcal{U}}(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) := (s, \mathbf{x}, \boldsymbol{E} - \lambda \boldsymbol{B}, \boldsymbol{B}). \tag{7.1.82}$$

Notice in contrast with the Lorentzian case, here there's no rotation in the $(\boldsymbol{E}, \boldsymbol{B})$ pair but rather a sum. This means duality transformations are not a symmetry of the equations of motion[6].

7.1.2 Electric limit

Maxwell theory admits two Carrollian limits, one of them being the already shown magnetic one, characterized by the magnetic field not transforming under Carrollian boosts. The remaining one is the electric Carrollian limit, which will be developed in what follows.

We start from the Carrollyfied Maxwell equations

[6]This is also a known result, duality transformations as understood in Lorentzian geometry act as maps between the electric and magnetic limits in both Carrollian and Galilean electrodynamics Duval et al. (2014c)

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial s} = 0 \qquad \nabla \cdot \boldsymbol{E} = 0 \tag{7.1.83}$$

$$-\frac{1}{C^2}\nabla \times \boldsymbol{B} + \frac{\partial \boldsymbol{E}}{\partial s} = 0 \qquad \nabla \cdot \boldsymbol{B} = 0. \tag{7.1.84}$$

And simply take the limit $C \to \infty$ so we arrive at the electric limit of Carrollian electromagnetism

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial s} = 0 \qquad \nabla \cdot \boldsymbol{E} = 0 \tag{7.1.85}$$

$$\frac{\partial \boldsymbol{E}}{\partial s} = 0 \qquad \nabla \cdot \boldsymbol{B} = 0. \tag{7.1.86}$$

This system of equations of motion is invariant under action of the flat Carrollian group, namely time translations, space translations, space rotations and Carrollian boosts. Time and space translations do not change the electric and magnetic field, rotations act in the expected and usual way and Carrollian boosts act as[7]

$$\boldsymbol{E}_{\boldsymbol{e}}(\boldsymbol{x}, s) \to \boldsymbol{E}'_{\boldsymbol{e}}(\boldsymbol{x}, s) = \boldsymbol{E}_{\boldsymbol{e}}(\boldsymbol{x}, s - \boldsymbol{b}\cdot\boldsymbol{x}) \tag{7.1.87}$$

$$\boldsymbol{B}_{\boldsymbol{e}}(\boldsymbol{x}, s) \to \boldsymbol{B}'_{\boldsymbol{e}}(\boldsymbol{x}, s) = \boldsymbol{B}_{\boldsymbol{e}}(\boldsymbol{x}, s - \boldsymbol{b}\cdot\boldsymbol{x}) - \boldsymbol{b} \times \boldsymbol{E}_{\boldsymbol{e}}(\boldsymbol{x}, s - \boldsymbol{b}\cdot\boldsymbol{x}). \tag{7.1.88}$$

7.1.2.1 Symmetries of the electric limit

This limit is expected to have the same symmetries as its magnetic counterpart but with slightly different actions resulting from the difference in sign in one relevant equation[8]. To find precisely said actions the Lie point symmetry method was employed. Let $(\mathcal{E}_e, \pi_e, C^{3+1})$ be the fiber bundle with base space C^{3+1} and projection map

[7] It was mentioned before but a good way of remembering which limit is which is thinking of what field does not transform under boosts.

[8] Recall, also, that duality transformations swap between the limits.

$$\begin{aligned} \pi_e : \mathcal{E}_e &\longrightarrow C^{3+1} && (7.1.89) \\ (s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) &\longrightarrow \pi_e \left(s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}\right) := \left(s, \mathbf{x}\right), && (7.1.90) \end{aligned}$$

whose tangent bundle is used to construct the equations of motion. Just as in the previous case, we obtained a system of highly over-determined partial differential equations which were solved polynomially. These polynomial solutions corresponds to components of vector fields that serve as generators of symmetries of the equations of motion from which they were constructed.

As we already knew, space translations are a symmetry of this limit

$$\mathcal{P}_A = \frac{\partial}{\partial x^A}. \tag{7.1.91}$$

Solving the following system of ODEs

$$\dot{\gamma}(\lambda) = \mathcal{P}_{A\,\gamma(\lambda)}, \tag{7.1.92}$$

the flows are constructed

$$\begin{aligned} h^{\mathcal{P}_1} : \mathbb{R} \times \mathcal{E}_e &\longrightarrow \mathcal{E}_e && (7.1.93) \\ (\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) &\longrightarrow h^{\mathcal{P}_1} \left(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}\right) := \left(s, x + \lambda, y, z, \boldsymbol{E}, \boldsymbol{B}\right) && (7.1.94) \\ h^{\mathcal{P}_2} : \mathbb{R} \times \mathcal{E}_e &\longrightarrow \mathcal{E}_e && (7.1.95) \\ (\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) &\longrightarrow h^{\mathcal{P}_2} \left(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}\right) := \left(s, x, y + \lambda, z, \boldsymbol{E}, \boldsymbol{B}\right) && (7.1.96) \\ h^{\mathcal{P}_2} : \mathbb{R} \times \mathcal{E}_e &\longrightarrow \mathcal{E}_e && (7.1.97) \\ (\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) &\longrightarrow h^{\mathcal{P}_3} \left(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}\right) := \left(s, x, y, z + \lambda, \boldsymbol{E}, \boldsymbol{B}\right), && (7.1.98) \end{aligned}$$

which can be written in a more compact way as

$$h^{\mathcal{P}} : \mathbb{R}^3 \times \mathcal{E}_e \longrightarrow \mathcal{E}_e \tag{7.1.99}$$

$$(\boldsymbol{\lambda}, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) \longrightarrow h^{\mathcal{P}}(\boldsymbol{\lambda}, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) := (s, \mathbf{x} + \boldsymbol{\lambda}, \boldsymbol{E}, \boldsymbol{B}) . \tag{7.1.100}$$

The time translations generator $\mathcal{H} \in T\mathcal{E}_e$ is the same as in the magnetic limit

$$\mathcal{H} = \frac{\partial}{\partial s}. \tag{7.1.101}$$

That means the system of ODEs to solve is the same

$$\dot{\gamma}(\lambda) = \mathcal{H}_{\gamma(\lambda)}, \tag{7.1.102}$$

and the flow constructed from its solutions are also the same

$$h^{\mathcal{H}} : \mathbb{R} \times \mathcal{E}_e \longrightarrow \mathcal{E}_e \tag{7.1.103}$$

$$(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) \longrightarrow h^{\mathcal{H}}(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) := (s + \lambda, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) . \tag{7.1.104}$$

This is also the case for all three generators of rotations $\mathcal{J}_A \in T\mathcal{E}_e$, which has to serve as a consistency check. For had this not been the case, there would have been at least one vector field[9] that would have transformed wrongly

$$\mathcal{J}_I = \epsilon_{IJK}\left(x^J \frac{\partial}{\partial x^K} + E^J \frac{\partial}{\partial E_K} + B^J \frac{\partial}{\partial B_K}\right). \tag{7.1.105}$$

The system of ordinary differential equations to solve is

[9]In the vector calculus sense.

$$\dot{\gamma}(\lambda) = \mathcal{J}_{A\,\gamma(\lambda)}. \tag{7.1.106}$$

Solutions of this system are used to build the appropriate symmetry transformations as flows

$$h^{\mathcal{J}_A} : \mathbb{R} \times \mathcal{E}_e \longrightarrow \mathcal{E}_e \tag{7.1.107}$$

$$(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) \longrightarrow h^{\mathcal{J}_A}(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}), \tag{7.1.108}$$

with the first being a rotation of angle λ with respect to the x-axis

$$\begin{aligned} h^{\mathcal{J}_1}(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) := \Big(& s, x, y\cos\lambda + z\sin\lambda, z\cos\lambda - y\sin\lambda, \\ & E_1, E_2\cos\lambda + E_3\sin\lambda, E_3\cos\lambda - E_2\sin\lambda, \\ & B_1, B_2\cos\lambda + B_3\sin\lambda, B_3\cos\lambda - B_2\sin\lambda\Big), \end{aligned} \tag{7.1.109}$$

the second being a rotation of angle λ with respect to the y-axis

$$\begin{aligned} h^{\mathcal{J}_2}(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) := \Big(& s, x\cos\lambda - z\sin\lambda, y, z\cos\lambda + x\sin\lambda, \\ & E_1\cos\lambda - E_3\sin\lambda, E_2, E_3\cos\lambda + E_1\sin\lambda, \\ & B_1\cos\lambda - B_3\sin\lambda, B_2, B_3\cos\lambda + B_1\sin\lambda\Big), \end{aligned} \tag{7.1.110}$$

and the third being a rotation of angle λ with respect to the z-axis

$$\begin{aligned} h^{\mathcal{J}_3}(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) := \Big(& s, x\cos\lambda + y\sin\lambda, y\cos\lambda - x\sin\lambda, z, \\ & E_1\cos\lambda + E_2\sin\lambda, E_2\cos\lambda - E_1\sin\lambda, E_3, \\ & B_1\cos\lambda + B_2\sin\lambda, B_2\cos\lambda - B_1\sin\lambda, B_3\Big). \end{aligned} \tag{7.1.111}$$

Perhaps it would be convenient to write them in term of rotation matrices, so let us define the three $SO(3)$ rotation matrices $R_A(\lambda)$

$$R_1(\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\lambda & -\sin\lambda \\ 0 & \sin\lambda & \cos\lambda \end{pmatrix} \quad R_2(\lambda) = \begin{pmatrix} \cos\lambda & 0 & \sin\lambda \\ 0 & 1 & 0 \\ -\sin\lambda & 0 & \cos\lambda \end{pmatrix} \quad R_3(\lambda) = \begin{pmatrix} \cos\lambda & -\sin\lambda & 0 \\ \sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{7.1.112}$$

this allows us to write the flows $h^{\mathcal{J}_A}$ in a more compact way as

$$h^{\mathcal{J}_3}\left(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}\right) = \left(s, R_A(\lambda)\mathbf{x}, R_A(\lambda)\boldsymbol{E}, R_A(\lambda)\boldsymbol{B}\right). \tag{7.1.113}$$

Carrollian supertranslations of the electric limit of Maxwell's equations have a three-parameter generator

$$\mathcal{T}_{abc} = x^a y^b z^c \frac{\partial}{\partial s} + \epsilon_{IJK} \frac{\partial\left(x^a y^b z^c\right)}{\partial x^I} E^J \frac{\partial}{\partial B_K}, \tag{7.1.114}$$

with $a, b, c \in \mathbb{N}$. The system of ODEs to solve in order to find the action of these generators is

$$\dot{\gamma}\left(\lambda\right) = \mathcal{T}_{abc\,\gamma(\lambda)}. \tag{7.1.115}$$

The flow of each $\mathcal{T}_{abc}$ is a 1-parameter transformation given by

$$h^{\mathcal{T}_{abc}} : \mathbb{R} \times \mathcal{E}_e \longrightarrow \mathcal{E}_e \tag{7.1.116}$$

$$\left(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}\right) \longrightarrow h^{\mathcal{T}_{abc}}\left(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}\right) := \left(s + \lambda x^a y^b z^c, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B} + \lambda \nabla\left(x^a y^b z^c\right) \times \boldsymbol{E}\right). \tag{7.1.117}$$

Taking the successive application of these transformations for different values of a, b and c we get a power-series expansion of an arbitrary $C^{\infty}(\mathbb{R}^3)$ function

$$f(x,y,z) = \sum_{a,b,c\in\mathbb{N}_0} \lambda_{abc} x^a y^b z^c. \tag{7.1.118}$$

Using this, it is possible to summarize these transformations as

$$h^{\mathcal{T}} : C^{\infty}\left(\mathbb{R}^3\right) \times \mathcal{E}_e \longrightarrow \mathcal{E}_e \tag{7.1.119}$$

$$(f, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) \longrightarrow h^{\mathcal{M}}(f, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) := (s + f(x,y,z), \mathbf{x}, \boldsymbol{E}, \boldsymbol{B} + \nabla f \times \boldsymbol{E}). \tag{7.1.120}$$

Supertranslations in the electric limit, just as in its magnetic counterpart, have Carrollian boosts as a subgroup which is recovered by considering only linear $C^{\infty}(\mathbb{R}^3)$-functions and the action of them yields the already-known transformations under boosts[10].

Next in line are time dilations $\mathcal{D} \in T\mathcal{E}_e$. In contrast to the magnetic limit's spatial dilation, there's a difference in sign for the magnetic field part

$$\mathcal{D} = x^I \frac{\partial}{\partial x^I} - B_A \frac{\partial}{\partial B_A}. \tag{7.1.121}$$

The system of ODEs to solve in order to find how this transformation act is

$$\dot{\gamma}(\lambda) = \mathcal{D}_{\gamma(\lambda)}. \tag{7.1.122}$$

Solutions to this system of differential equations are used to construct the flow

[10] Same considerations as previously must be taken.

$$h^{\mathcal{D}} : \mathbb{R} \times \mathcal{E}_e \longrightarrow \mathcal{E}_e \tag{7.1.123}$$
$$(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) \longrightarrow h^{\mathcal{D}} (\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) := \left(s, e^{\lambda}\mathbf{x}, \boldsymbol{E}, e^{-\lambda}\boldsymbol{B}\right). \tag{7.1.124}$$

Time dilations $\mathcal{Q} \in T\mathcal{E}_e$ also have a different sign in the magnetic field transformation part if we compare it with its counterpart in the magnetic limit

$$\mathcal{Q} = s\frac{\partial}{\partial s} + B_A \frac{\partial}{\partial B_A}. \tag{7.1.125}$$

The system of ODEs to solve to find the symmetry transformation for time dilations is

$$\dot{\gamma} (\lambda) = \mathcal{Q}_{\gamma(\lambda)}. \tag{7.1.126}$$

Solutions to this system are used to construct the appropriate flow

$$h^{\mathcal{Q}} : \mathbb{R} \times \mathcal{E}_e \longrightarrow \mathcal{E}_e \tag{7.1.127}$$
$$(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) \longrightarrow h^{\mathcal{Q}} (\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) := \left(e^{\lambda}s, \mathbf{x}, \boldsymbol{E}, e^{\lambda}\boldsymbol{B}\right). \tag{7.1.128}$$

Both spatial and time dilations correspond to actions of the multiplicative group $(\mathbb{R}^{\times}, \cdot)$.

Special conformal transformations in the Carrollian electric limit are characterized by vectors $\mathcal{S}_A \in T\mathcal{E}_e$ given by

$$\mathcal{S}_A = 2x_A\left(x^B\frac{\partial}{\partial x^B} + s\frac{\partial}{\partial s}\right) - x_B x^B\frac{\partial}{\partial x^A} \tag{7.1.129}$$

$$- 4x_A E_J\frac{\partial}{\partial E^J} + 2x^J\left(E_A\frac{\partial}{\partial E_J} - E_J\frac{\partial}{\partial E^A}\right) \tag{7.1.130}$$

$$- 4x_A B_J\frac{\partial}{\partial B_J} + 2x^J\left(B_A\frac{\partial}{\partial B_J} - B_J\frac{\partial}{\partial B^A}\right) + 2s\,\epsilon_{AJK}E^J\frac{\partial}{\partial B_K}. \tag{7.1.131}$$

The system of ODEs to solve in order to find how these transformations act is

$$\dot{\gamma}(\lambda) = \mathcal{S}_{A\,\gamma(\lambda)}. \tag{7.1.132}$$

Solutions to these equations are used to construct the flows that correspond to each special conformal transformation

$$h^{\mathcal{S}_A} : \mathbb{R}\times\mathcal{E}_e \longrightarrow \mathcal{E}_e \tag{7.1.133}$$

$$(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) \longrightarrow h^{\mathcal{S}_A}(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}). \tag{7.1.134}$$

Explicitly we have

$$h^{\mathcal{S}_1}(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) = (\omega_x(\lambda)s, \omega_x(\lambda)(x - \lambda\mathbf{x}\cdot\mathbf{x}), \omega_x(\lambda)y, \omega_x(\lambda)z, \mathbb{T}_1(\lambda)\boldsymbol{E}, \mathbb{T}_1(\lambda)\boldsymbol{B} - \mathbb{O}_1(\lambda)\boldsymbol{E}) \tag{7.1.135}$$

$$h^{\mathcal{S}_2}(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) = (\omega_y(\lambda)s, \omega_y(\lambda)x, \omega_y(\lambda)(y - \lambda\mathbf{x}\cdot\mathbf{x}), \omega_y(\lambda)z, \mathbb{T}_2(\lambda)\boldsymbol{E}, \mathbb{T}_2(\lambda)\boldsymbol{B} - \mathbb{O}_2(\lambda)\boldsymbol{E}) \tag{7.1.136}$$

$$h^{\mathcal{S}_3}(\lambda, s, \mathbf{x}, \boldsymbol{E}, \boldsymbol{B}) = (\omega_z(\lambda)s, \omega_z(\lambda)x, \omega_z(\lambda)y, \omega_z(\lambda)(z - \lambda\mathbf{x}\cdot\mathbf{x}), \mathbb{T}_3(\lambda)\boldsymbol{E}, \mathbb{T}_3(\lambda)\boldsymbol{B} - \mathbb{O}_3(\lambda)\boldsymbol{E}), \tag{7.1.137}$$

where $\mathbb{T}_A(\lambda)$ and $\mathbb{O}_A(\lambda)$ were defined in the previous section. Notice there is a sign change in how $\mathbb{O}_A(\lambda)$ enters the transformation.

7.1.3 Space-time symmetries: the algebra

Both limits have their own set of vector fields generating their symmetries which differ only in how they transform the electric and magnetic field. By taking the pushforward of the projection map[11] we get the spatial part of said vector fields. Recall

$$(\pi_m)_* : T\mathcal{E}_m \longrightarrow TC^{3+1} \qquad (\pi_e)_* : T\mathcal{E}_e \longrightarrow TC^{3+1} \tag{7.1.138}$$

$$\mathcal{X} \longrightarrow (\pi_m)_* \mathcal{X} \qquad \mathcal{X} \longrightarrow (\pi_e)_* \mathcal{X}. \tag{7.1.139}$$

For simplicity we write π_* as a stand-in for the appropriate pushforward

$$P_I = \pi_* \mathcal{P}_I = \frac{\partial}{\partial x^I} \tag{7.1.140}$$

$$H = \pi_* \mathcal{H} = \frac{\partial}{\partial s} \tag{7.1.141}$$

$$J_I = \pi_* \mathcal{J}_I = \epsilon_{IJK} x^J \frac{\partial}{\partial x^K} \tag{7.1.142}$$

$$T_{nmj} = \pi_* \mathcal{T}_{nmj} = x^n y^m z^k \frac{\partial}{\partial s} \tag{7.1.143}$$

$$D = \pi_* \mathcal{D} = x^I \frac{\partial}{\partial x^I} \tag{7.1.144}$$

$$Q = \pi_* \mathcal{Q} = s \frac{\partial}{\partial s} \tag{7.1.145}$$

$$S_A = \pi_* \mathcal{S}_A = 2x_A \left(x^B \frac{\partial}{\partial x^B} + s \frac{\partial}{\partial s} \right) - x_B x^B \frac{\partial}{\partial x^A}. \tag{7.1.146}$$

Let V be the real vector space spanned by the vector fields defined above

$$V = \operatorname{span}_{\mathbb{R}} \{P_A, H, J_A, T_{abc}, D, Q, S_A\}. \tag{7.1.147}$$

With $A \in \{1, 2, 3\}$ and $a, b, c \in \mathbb{N}_0$. The vector space V, together with the differential-geometric commutator $[\cdot, \cdot]$ form an infinite-dimensional algebra

[11]Either $(\pi_m)_*$ or $(\pi_e)_*$ depending on whether we are working with $\mathcal{E}_m$ or $\mathcal{E}_e$, respectively.

$(V, [\cdot,\cdot])$. This possesses finite sub-algebra which corresponds to the Carrollian algebra. We take

$$K_1 = T_{100} = x\frac{\partial}{\partial s} \qquad K_2 = T_{010} = y\frac{\partial}{\partial s} \qquad K_3 = T_{001} = z\frac{\partial}{\partial s}. \tag{7.1.148}$$

Their commutators are taken to be the differential-geometric Lie Bracket

$$[P_A, P_B] = 0 \qquad [P_A, H] = 0 \qquad [P_A, J_B] = \epsilon_{ABC} J_K \tag{7.1.149}$$

$$[P_A, K_B] = \delta_{AB} H \qquad [H, J_A] = 0 \qquad [H, K_A] = 0 \tag{7.1.150}$$

$$[J_A, J_B] = \epsilon_{ABC} J_C \qquad [J_A, K_B] = \epsilon_{ABC} K_C \qquad [K_A, K_B] = 0 \tag{7.1.151}$$

It can be seen that this subalgebra closes and corresponds to the Carrollian Lie algebra. This was, of course, to be expected as previous works had already proven it. The rest of the commutator table is the following

$$[T_{qwe}, T_{rty}] = 0 \qquad [T_{nmj}, Q] = T_{nmj} \qquad [T_{nmj}, H] = 0 \tag{7.1.152}$$

$$[T_{nmj}, P_1] = -nT_{n-1\,mj} \quad [T_{nmj}, P_2] = -mT_{n\,m-1\,j} \qquad [T_{nmj}, P_3] = -jT_{nm\,j-1} \tag{7.1.153}$$

$$[H, D] = 0 \qquad [D, T_{nmj}] = (n+m+j)\,T_{nmj} \qquad [P_A, S_B] = 2\delta_{AB}\,(D+Q) \tag{7.1.154}$$

$$[H, S_A] = 2K_A \qquad [D, S_I] = S_I \qquad [D, Q] = 0 \tag{7.1.155}$$

$$[D, J_A] = 0 \qquad [P_A, D] = P_A \qquad [S_A, S_B] = 0 \tag{7.1.156}$$

$$[P_A, Q] = 0 \qquad [H, Q] = H \qquad [J_A, Q] = 0 \tag{7.1.157}$$

$$[J_A, S_B] = \epsilon_{ABC} S_C \qquad [Q, S_A] = 0, \tag{7.1.158}$$

and the ones that were far too long to be included above

$$[T_{nmj}, S_1] = (2 - 2j - 2m - n)\, T_{n+1\,mj} + n\,(T_{n-1\,m+2\,j} + T_{n-1\,m\,j+2}) \quad (7.1.159)$$

$$[T_{nmj}, S_2] = (2 - 2j - m - 2n)\, T_{n\,m+1\,j} + m\,(T_{n+2\,m-1\,j} + T_{n\,m-1\,j+2}) \quad (7.1.160)$$

$$[T_{nmj}, S_3] = (2 - j - 2m - 2n)\, T_{nm\,j+1} + j\,(T_{n\,m+2\,j-1} + T_{n+2\,m\,j-1}) \quad (7.1.161)$$

$$[J_1, T_{nmj}] = mT_{n\,m-1\,j+1} - jT_{n\,m+1\,j-1} \quad (7.1.162)$$

$$[J_2, T_{nmj}] = jT_{n+1\,m\,j-1} - nT_{n-1\,m\,j+1} \quad (7.1.163)$$

$$[J_3, T_{nmj}] = nT_{n-1\,m+1\,j} - mT_{n+1\,m-1\,j}. \quad (7.1.164)$$

Notice the downward ladder is truncated at zero for the index values of T_{nmj}. This means that the range of $\{n, m, j\}$ is $\mathbb{N}_0^3$. However, negative values can be included and the algebra still closes[12].

This is all well and good but characterization is needed in order to be able to properly talk about this group. We found that this has an overlap with the Conformal Carrollian algebra of order 2. Conformal Carroll groups of order k are vector fields X which satisfy the condition according to Duval et al. (2014a)

$$L_X \left(g \otimes \xi^{\otimes k}\right) = 0, \quad (7.1.165)$$

where $g = \delta_{AB} dx^A \otimes dx^B$ and $\xi^{\otimes k} = \bigotimes_{n=1}^{k} \xi$. The vector fields obtained above satisfy this criterion for $k = 2$. We first get (7.1.165) into a readier expression to compute

$$L_X \left(g \otimes \xi^{\otimes 2}\right) = \delta_{AB} \left(L_X dx^A\right) \otimes dx^B \otimes \xi^{\otimes 2} + \delta_{AB} dx^A \otimes \left(L_X dx^B\right) \otimes \xi^{\otimes 2}$$
$$+ g \otimes (L_X \xi) \otimes \xi + g \otimes \xi \otimes (L_X \xi)\,, \quad (7.1.166)$$

which means we have to compute two different kind of terms, namely

[12] So far, negative exponents have been absent in the discussion because our approach excludes them in the polynomial expansions.

$$L_X dx^A = d\left(dx^A(X)\right) \qquad L_X\xi = [X,\xi]\,. \tag{7.1.167}$$

We start by calculating for the three generators of spatial translations, for which the Lie derivatives yields zero by virtue of one being the exterior derivative of an exact form and the other one by having null commutator

$$L_{P_I} dx^A = d\left(dx^A\left(\frac{\partial}{\partial x^I}\right)\right) \qquad L_{P_I}\xi = [P_I,\xi] \tag{7.1.168}$$

$$= 0 \qquad = 0. \tag{7.1.169}$$

For the generator of time translations we have the exact same picture as before. It is worthy of mention that under our definitions $H = \xi$

$$L_H dx^A = d\left(dx^A(H)\right) \qquad L_H\xi = [H,\xi] \tag{7.1.170}$$

$$= 0 \qquad = 0. \tag{7.1.171}$$

For the generators of spatial rotations the Lie derivative of the metric g is zero as shown in chapter 2. The commutator in this case also vanishes

$$L_{J_I} dx^A = d\left(dx^A(J_I)\right) \qquad L_{J_I}\xi = [J_I,\xi] \tag{7.1.172}$$

$$= d\left(\epsilon_{IJA}x^J\right) \qquad = 0 \tag{7.1.173}$$

$$= \epsilon_{IJA}dx^J. \tag{7.1.174}$$

We have $L_{J_I}g$ is zero on accounts of being the symmetrization of an antisymmetric object, as shown before. Super-translations are in the kernel of dx^A and also have zero commutator with ξ

$$L_{T_{abc}} dx^A = d\left(dx^A\left(x^a y^b z^c \frac{\partial}{\partial s}\right)\right) \qquad L_{T_{abc}} = [T_{abc}, \xi] \tag{7.1.175}$$

$$= 0 \qquad = 0. \tag{7.1.176}$$

Spatial dilations do not satisfy the criteria for being part of the Carrollian conformal algebra of order two, as seen in the following

$$L_D dx^A = d\left(dx^A\left(x^I \frac{\partial}{\partial x^I}\right)\right) \qquad L_D \xi = [D, \xi] \tag{7.1.177}$$

$$= dx^A \qquad = 0. \tag{7.1.178}$$

This yields $L_D\left(g \otimes \xi^{\otimes 2}\right) = 2g \otimes \xi^{\otimes 2}$ and not zero. A quick conclusion from this is the algebra spanned by these generators is not a subalgebra of the one we are interested in. However, this can be compensated with the second-to-last generator

$$L_Q dx^A = d\left(dx^A\left(s \frac{\partial}{\partial s}\right)\right) \qquad L_Q \xi = [Q, \xi] \tag{7.1.179}$$

$$= 0 \qquad = -\xi. \tag{7.1.180}$$

This yields $L_Q\left(g \otimes \xi^{\otimes 2}\right) = -2g \otimes \xi^{\otimes 2}$ and also not zero[13]. Therefore, it is not a member of the conformal Carrollian algebra. On their own, neither D nor Q constitute members of this algebra. However, their sum $Y = D + Q$ is since $L_Y\left(g \otimes \xi^{\otimes 2}\right) = L_{Q+D}\left(g \otimes \xi^{\otimes 2}\right) = \left(L_D + L_Q\right)\left(g \otimes \xi^{\otimes 2}\right) = 0$.

Although we have been calling them special conformal transformations, it was at this point we found out exactly which kind of special conformal transformations they are, namely of level $k = 2$. We check it satisfies the criteria as follows

[13]Note had we chosen $k = 0$ this would have satisfied the criteria.

$$L_{S_A} dx^I = d\left(dx^I\left(2x_A\left(x^B\frac{\partial}{\partial x^B} + s\frac{\partial}{\partial s}\right) - x_B x^B\frac{\partial}{\partial x^A}\right)\right) \qquad L_{S_A}\xi = [S_A, \xi] \tag{7.1.181}$$

$$= 2x^I dx_A + 2x_A dx^I - 2\delta^I_A x_B dx^B \qquad = -2x_A\xi. \tag{7.1.182}$$

Using this we get $L_{S_A} g = 4x_A g$ and $L_{S_A}\left(g \otimes \xi^{\otimes 2}\right) = 0$. This is only possible since we chose $k = 2$. Whether there are more vectors in the conformal Carrollian algebra with $k = 2$ is beyond the scope of the present work.

In general, for (7.1.165) to be satisfied it is needed that

$$L_X g = \Omega g \qquad L_X \xi = -\frac{\Omega}{k}\xi. \tag{7.1.183}$$

Let $X = X^A \frac{\partial}{\partial x^A} + X^s \frac{\partial}{\partial s}$, then

$$L_X \xi = [X, \xi] \tag{7.1.184}$$

$$= -\frac{\partial X^A}{\partial s}\frac{\partial}{\partial x^A} - \frac{\partial X^s}{\partial s}\frac{\partial}{\partial s}. \tag{7.1.185}$$

It follows $\frac{\partial X^A}{\partial s} = 0$. $L_X dx^B = d\left(dx^B(X)\right) = d\left(R^B\right)$, then

$$L_X g = \delta_{BC}\left[\left(L_X dx^B\right) \otimes dx^C + dx^B \otimes \left(L_X dx^C\right)\right] \tag{7.1.186}$$

$$= \delta_{BC}\left[\frac{\partial X^B}{\partial x^A} dx^A \otimes dx^C + \frac{\partial X^C}{\partial x^A} dx^B \otimes dx^A\right] \tag{7.1.187}$$

$$= \left(\frac{\partial X^A}{\partial x^B} + \frac{\partial X^B}{\partial x^A}\right) dx^A \otimes dx^B. \tag{7.1.188}$$

Putting this together we get

$$\frac{\partial X^A}{\partial x^B} + \frac{\partial X^B}{\partial x^A} - k\,\delta_{AB}\frac{\partial X^s}{\partial s} = 0. \tag{7.1.189}$$

By taking the time derivative of this expression we conclude X^s is at most linear in time s.

7.1.4 Space-time symmetries: the finite transformations

The restriction to the space-time symmetry transformations of each limit is done by taking the projection map π_m or π_e for the magnetic and electric case, respectively. For any $X \in V$ and $\lambda \in \mathbb{R}$ the map h^X_λ is a spatio-temporal symmetry of these limits. A notorious simplification appears for CSCT after doing this, namely

$$h^{\boldsymbol{S}} : \mathbb{R}^3 \times C^{3+1} \longrightarrow C^{3+1} \tag{7.1.190}$$

$$(\boldsymbol{\lambda}, s, \boldsymbol{x}) \longrightarrow h^{\boldsymbol{S}}(\boldsymbol{\lambda}, s, \boldsymbol{x}), \tag{7.1.191}$$

where

$$h^{\boldsymbol{S}}(\boldsymbol{\lambda}, s, \boldsymbol{x}) := \left(\frac{\boldsymbol{x}\cdot\boldsymbol{x}\, s}{(\boldsymbol{x} - \boldsymbol{\lambda}\,\boldsymbol{x}\cdot\boldsymbol{x})\cdot(\boldsymbol{x} - \boldsymbol{\lambda}\,\boldsymbol{x}\cdot\boldsymbol{x})}, \frac{\boldsymbol{x}\cdot\boldsymbol{x}\,(\boldsymbol{x} - \boldsymbol{\lambda}\,\boldsymbol{x}\cdot\boldsymbol{x})}{(\boldsymbol{x} - \boldsymbol{\lambda}\,\boldsymbol{x}\cdot\boldsymbol{x})\cdot(\boldsymbol{x} - \boldsymbol{\lambda}\,\boldsymbol{x}\cdot\boldsymbol{x})} \right). \tag{7.1.192}$$

7.2 At the level of the Hamiltonian

The Hamiltonian approach is natural in both Galilean and Carroll geometries since they both carry a choice of time given by the clock form θ and the vector field ξ, respectively. These choices give us a canonical Hamiltonian foliation to work with.

Gauge theories have two pairs of equations of motion, one coming from a Lagrangian density $\mathcal{L}$ and one from a Bianchi identity

$$d_A F = 0. \tag{7.2.1}$$

Because of this, Hamiltonian descriptions of gauge theories rely on Lagrange multipliers in order to account for the fact that not the entirety of variations of canonical variables in phase space is independent. An explicit choice of such Lagrange multipliers corresponds to a gauge fixing, as shown by Dirac in Dirac (2001).

The Hamiltonian description of Maxwell theory was used in Henneaux and Salgado-Rebolledo (2021) to obtain both Carrollian limits of electrodynamics. They also showed it works for gauge theories of the Yang-Mills type. Hamiltonian descriptions of electrodynamics are also discussed in classical textbooks such as Jackson (1999) and Zangwill (2013).

Direct canonical analysis yields the energy function

$$\mathcal{E} = \frac{1}{2}\left(c^2\pi^a\pi_a + \frac{1}{2}F_{ab}F^{ab}\right), \tag{7.2.2}$$

where $\pi^a := \dfrac{\partial\mathcal{L}}{\partial(\partial_t A_a)}$. The Lagrange multiplier $A_t\partial_a\pi^a$ to ensure charge conservation is added to construct the Hamiltonian

$$H = \int_V \left(\frac{1}{2}c^2\pi^a\pi_a + \frac{1}{4}F_{ab}F^{ab} - A_t\partial_a\pi^a\right) d^3\mathbf{x}. \tag{7.2.3}$$

Notice that this does not require a full space-time metric but only a spatial metric, used for both $F_{ab}F^{ab}$ and the integration measure. Said spatial measure can always be obtained from the restriction of the Lorentzian one to space by choosing time in accordance to ξ.

7.2.1 Magnetic limit

The magnetic limit is directly obtained by taking the limit $c \to 0$ in (7.2.3). This yields

$$H^M = \int_V \left(\frac{1}{4} F_{ab} F^{ab} - A_t \partial_a \pi^a \right) d^3\mathbf{x}. \tag{7.2.4}$$

Hamilton's equation of motion combined with Bianchi's identity yield the correct magnetic limit for Maxwell's electrodynamics described previously.

7.2.2 Electric limit

The electric limit is obtained from (7.2.3) after field reparametrization

$$A_a \to c A_a \qquad A_t \to c A_t \qquad \pi^a \to \frac{1}{c} \pi^a, \tag{7.2.5}$$

and then taking the limit $c \to 0$. Obtaining the electric Hamiltonian

$$H^E = \int_V \left(\frac{1}{2} \pi_a \pi^a - A_t \partial_a \pi^a \right) d^3\mathbf{x}. \tag{7.2.6}$$

Hamilton's equation of motion combined with Bianchi's identity yield the correct electric limit for Maxwell's electrodynamics described previously.

Part III

ModMax theory, symmetries and limits

Chapter 8

ModMax theory

8.1 Lagrangian formulation

Modified Maxwell theory, or ModMax for short, is the unique non-linear theory of electromagnetism that has the same symmetries as Maxwell. That is, it is a Lorentz-invariant, conformal and duality invariant in vacuum, $U(1)$-gauge field theory. It is defined by two pairs of equations, the first being the Bianchi identity

$$dF = 0, \tag{8.1.1}$$

which in its vector calculus form corresponds to the pair

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0. \tag{8.1.2}$$

The second pair of equations of ModMax theory comes from its Lagrangian, which is formulated in terms of two Lorentz invariant quantities[1] built using both the $U(1)$ curvature F and its Hodge dual $\bar{F} = \star F$ and are as follows

[1]The use of the word scalar was avoided here since only S is one. P is a pseudo-scalar.

$$S = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \qquad\qquad P = -\frac{1}{4}\bar{F}^{\mu\nu}F_{\mu\nu} \tag{8.1.3}$$

$$= \frac{1}{2}\left(\frac{E^2}{c^2} - B^2\right) \qquad\qquad = \frac{1}{c}\boldsymbol{B}\cdot\boldsymbol{E}. \tag{8.1.4}$$

With this, the Lagrangian of Maxwell free theory can be written quite simply as

$$\mathcal{L} = S. \tag{8.1.5}$$

In turn, ModMax theory is a 1-parameter family of Lagrangians given by

$$\mathcal{L}_\gamma := \cosh\gamma\, S + \sinh\gamma\sqrt{S^2 + P^2} \tag{8.1.6}$$

$$= \frac{1}{2}\cosh\gamma\left(\frac{E^2}{c^2} - B^2\right) + \sinh\gamma\sqrt{\frac{1}{4}\left(\frac{E^2}{c^2} - B^2\right)^2 + \frac{1}{c^2}\left(\boldsymbol{E}\cdot\boldsymbol{B}\right)^2}, \tag{8.1.7}$$

where $\gamma \in \mathbb{R}_0^+$ is a positive number to ensure that the energy has a lower bound in the quantum case, and the functions cosh and sinh were chosen to ensure dual invariance, as will be seen later. From this definition we can note that Maxwell theory is recovered when $\gamma = 0$ is chosen.

So far in this work I have refused to derive the EOM in term of coordinates. At this point, however, proceeding in that fashion can prove be a little more effort than it's worth.

The equations of motion coming from the ModMax Lagrangian are Banerjee and Mehra (2022)

$$
\begin{aligned}
0 =&\partial_\mu \left[\frac{\partial \mathcal{L}_\gamma}{\partial F_{\mu\nu}}\right]\\
=&\partial_\mu \left[\frac{\partial \mathcal{L}_\gamma}{\partial S}\frac{\partial S}{\partial F_{\mu\nu}} + \frac{\partial \mathcal{L}_\gamma}{\partial P}\frac{\partial P}{\partial F_{\mu\nu}}\right]\\
=&\partial_\mu \left[\cosh\gamma\, F^{\mu\nu} + \sinh\gamma \frac{S\, F^{\mu\nu} + P\, \bar{F}^{\mu\nu}}{\sqrt{S^2+P^2}}\right]
\end{aligned}
\tag{8.1.8}
$$

$$
\begin{aligned}
=&\cosh\gamma\, \partial_\mu F^{\mu\nu} + \sinh\gamma \frac{\partial_\mu \left(S\, F^{\mu\nu} + P\, \bar{F}^{\mu\nu}\right)\sqrt{S^2+P^2}}{S^2+P^2}\\
&- \sinh\gamma \frac{\left(S\, F^{\mu\nu} + P\, \bar{F}^{\mu\nu}\right)\left(S^2+P^2\right)^{-1/2}\left(S\partial_\mu S + P\partial_\mu P\right)}{S^2+P^2}\\
=&\cosh\gamma\, \partial_\mu F^{\mu\nu} + \sinh\gamma\, \frac{\left(S^2+P^2\right)\left(S\partial_\mu F^{\mu\nu} + F^{\mu\nu}\partial_\mu S + P\partial_\mu \bar{F}^{\mu\nu} + \bar{F}^{\mu\nu}\partial_\mu P\right)}{\left(S^2+P^2\right)^{3/2}}\\
&- \sinh\gamma \frac{S^2\partial_\mu S\, F^{\mu\nu} + P^2\partial_\mu P\, \bar{F}^{\mu\nu} + SP\partial_\mu P\, F^{\mu\nu} - SP\partial_\mu S\, \bar{F}^{\mu\nu}}{\left(S^2+P^2\right)^{3/2}}\\
=&\cosh\gamma\partial_\mu F^{\mu\nu} + \sinh\gamma \frac{\left(S^3+SP^2\right)\partial_\mu F^{\mu\nu} + \left(P^3+PS^2\right)\partial_\mu \bar{F}^{\mu\nu}}{\left(S^2+P^2\right)^{3/2}}\\
&+ \sinh\gamma\, \frac{S^2\bar{F}^{\mu\nu}\partial_\mu P + P^2 F^{\mu\nu}\partial_\mu S - SP\left(\partial_\mu P\, F^{\mu\nu} + \partial_\mu S\, \bar{F}^{\mu\nu}\right)}{\left(S^2+P^2\right)^{3/2}}\\
=&\cosh\gamma\partial_\mu F^{\mu\nu} + \sinh\gamma\, \left[\frac{S\partial_\mu F^{\mu\nu} + P\partial_\mu \bar{F}^{\mu\nu}}{\sqrt{S^2+P^2}}\right.\\
&\left. - \frac{\left(S^2\bar{F}^{\mu\nu} - SPF^{\mu\nu}\right)\partial_\mu P + \left(P^2F^{\mu\nu} - SP\bar{F}^{\mu\nu}\right)\partial_\mu S}{\left(S^2+P^2\right)^{3/2}}\right].
\end{aligned}
\tag{8.1.9}
$$

Where we have included some detailed calculations for future reference. While having the EOM written as in (8.1.9) will be the crucial to taking the Carrollian limits, it is useful to have them written in a slightly different way. To achieve this we refer to (8.1.8) and note that it has the shape of an exterior derivative of some form G. Our objective will be to find said G. First we rearrange the equation for it

$$
\partial_\mu \left[\left(\cosh\gamma + \sinh\gamma \frac{S}{\sqrt{S^2+P^2}}\right) F^{\mu\nu} + \sinh\gamma \frac{P}{\sqrt{S^2+P^2}}\bar{F}^{\mu\nu}\right] = 0. \tag{8.1.10}
$$

Remark:

$$\frac{\partial S}{\partial F_{\mu\nu}} = -\frac{1}{2}F^{\mu\nu} \qquad \frac{\partial P}{\partial F_{\mu\nu}} = -\frac{1}{2}\bar{F}^{\mu\nu}. \tag{8.1.11}$$

Equation (8.1.10) can be rewritten in terms of differential forms by using properties of the Levi-Civita symbol as

$$d\star G = 0, \tag{8.1.12}$$

where G is a 2-form given by

$$G = \left(\cosh\gamma + \sinh\gamma\frac{S}{\sqrt{S^2+P^2}}\right)F + \sinh\gamma\frac{P}{\sqrt{S^2+P^2}}\star F. \tag{8.1.13}$$

With this, ModMax equations of motion correspond to the pair

$$dF = 0 \qquad d\star G = 0, \tag{8.1.14}$$

which is quite reminiscing of Maxwell's equations written in differential forms. Writing them in this fashion is useful for quite different purposes, one of them being finding their conserved charges.

This also suggest the presence of duality invariance, which is the case. ModMax being duality invariant means it satisfies the Gaillard-Zumino criterion, first presented in Gaillard and Zumino (1981). In other words we have

$$(\star G)_{\mu\nu}G^{\mu\nu} = (\star F)_{\mu\nu}F^{\mu\nu}, \tag{8.1.15}$$

so duality transformations

$$\begin{pmatrix} G'_{\mu\nu} \\ (\star F)'_{\mu\nu} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} G_{\mu\nu} \\ (\star F)_{\mu\nu} \end{pmatrix} \tag{8.1.16}$$

$$= \begin{pmatrix} G_{\mu\nu}\cos\theta + (\star F)_{\mu\nu}\sin\theta \\ (\star F)_{\mu\nu}\cos\theta - G_{\mu\nu}\sin\theta \end{pmatrix} \tag{8.1.17}$$

leave ModMax invariant. Conformal invariance can be checked by noting the stress energy tensor of the theory is traceless.

8.2 Hamiltonian formulation

In terms of Hamiltonian formulation, we do not have Hamiltonian formulation.[2] In spite of not having been able to construct a proper Hamiltonian, we can build the energy density function in terms of the electric and magnetic fields. We start with the canonical momenta

$$\pi^a = -\frac{1}{c^2}\cosh\gamma\,\boldsymbol{E} - \frac{1}{2c^2}\sinh\gamma\,\frac{\left[\frac{E^2}{c^2} - B^2\right]\boldsymbol{E} + 2\left[\boldsymbol{E}\cdot\boldsymbol{B}\right]\boldsymbol{B}}{\sqrt{\frac{1}{4}\left(\frac{E^2}{c^2} - B^2\right)^2 + \frac{1}{c^2}\left(\boldsymbol{E}\cdot\boldsymbol{B}\right)^2}} \tag{8.2.1}$$

And, as should be the case, the canonical momentum π^0 associated to the scalar potential is $\pi^0 = 0$. With this, we can construct the energy density of the theory[3] in the usual way

$$H = \int_\Omega \left[\dot{\boldsymbol{A}}\cdot\boldsymbol{\pi} - \mathcal{L}\right] d^3\mathbf{x} \tag{8.2.2}$$

[2]Even though ModMax was constructed from its energy density function, we do not have its Hamiltonian formulation, for it requires us to be able to solve the time derivatives of the connection in terms of their canonical momenta. This task has proven difficult given the non-linear aspects of the theory.

[3]I need to emphasize this is *not* the Hamiltonian as it would need to be a function of phase space. If the expression we arrive at were to be written in term of canonical variables then it would be the Hamiltonian. Not before.

Here it is possible to exploit the fact that $\boldsymbol{E} = -\dot{\boldsymbol{A}} - \nabla\phi$ to write

$$H = \int_\Omega \left[-\boldsymbol{E}\cdot\boldsymbol{\pi} - \nabla\phi\cdot\boldsymbol{\pi} - \mathcal{L}\right] d^3\mathbf{x} \tag{8.2.3}$$

$$= \int \left[\frac{1}{c^2}\cosh\gamma\, E^2 + \frac{1}{2c^2}\sinh\gamma \frac{\left[\frac{E^2}{c^2} - B^2\right] E^2 + 2\left[\boldsymbol{E}\cdot\boldsymbol{B}\right]^2}{\sqrt{\frac{1}{4}\left(\frac{E^2}{c^2} - B^2\right)^2 + \frac{1}{c^2}\left(\boldsymbol{E}\cdot\boldsymbol{B}\right)^2}} - \nabla\phi\cdot\boldsymbol{\pi} - \mathcal{L}\right] d^3\mathbf{x} \tag{8.2.4}$$

$$= \int \left[\frac{1}{2}\cosh\gamma\, \left(\frac{E^2}{c^2} + B^2\right) + \frac{1}{4}\sinh\gamma \frac{\left(\frac{E^2}{c^2} - B^2\right)\left(\frac{E^2}{c^2} + B^2\right)}{\sqrt{\frac{1}{4}\left(\frac{E^2}{c^2} - B^2\right)^2 + \frac{1}{c^2}\left(\boldsymbol{E}\cdot\boldsymbol{B}\right)^2}} - \nabla\phi\cdot\boldsymbol{\pi}\right] d^3\mathbf{x} \tag{8.2.5}$$

Integrating by parts the last addend in the previous expression we arrive at the energy function

$$H = \int \left[\frac{1}{2}\cosh\gamma\, \left(\frac{E^2}{c^2} + B^2\right) + \frac{1}{4}\sinh\gamma \frac{\left(\frac{E^2}{c^2} - B^2\right)\left(\frac{E^2}{c^2} + B^2\right)}{\sqrt{\frac{1}{4}\left(\frac{E^2}{c^2} - B^2\right)^2 + \frac{1}{c^2}\left(\boldsymbol{E}\cdot\boldsymbol{B}\right)^2}} + \phi\nabla\cdot\boldsymbol{\pi}\right] d^3\mathbf{x}. \tag{8.2.6}$$

Although not a complete Hamiltonian formulation[4], this expression will still be useful in a following section to construct a Hamiltonian formulation of both the electric and magnetic Carrollian limit of ModMax theory.

[4]And I cannot stress this enough, since we do not have an explicit solution for the canonical momenta $\boldsymbol{\pi}$ this is not written in canonical variables and, therefore, is *not* the Hamiltonian.

Chapter 9

Carrollian limits

9.1 At the level of the equations of motion

From vacuum ModMax electrodynamics it is possible to construct two nonequivalent limits which are Carroll-covariant, namely the so-called electric and magnetic limit.

9.1.1 Electric limit

For the electric limit we re-scale $\boldsymbol{E_e} = \boldsymbol{E}$, $s = (cC)\,t$ and $\boldsymbol{B_e} = (cC)\,\boldsymbol{B}$ in Maxwell's equations, then take the limit $C \to \infty$.

$$\nabla \times \boldsymbol{E_e} + \frac{\partial \boldsymbol{B_e}}{\partial s} = 0 \qquad \nabla \cdot \boldsymbol{B_e} = 0 \tag{9.1.1}$$

$$(\cosh\gamma + \sinh\gamma)\,\frac{\partial \boldsymbol{E_e}}{\partial s} = 0 \qquad (\cosh\gamma + \sinh\gamma)\,\nabla \cdot \boldsymbol{E_e} = 0. \tag{9.1.2}$$

This electric limit is equivalent to its Maxwell counterpart and proves to be Carroll invariant, with transformations under boosts given by:

$$\boldsymbol{E_e}(\boldsymbol{x}, s) \to \boldsymbol{E'_e}(\boldsymbol{x}, s) = \boldsymbol{E_e}(\boldsymbol{x}, s - \boldsymbol{b}\cdot\boldsymbol{x}) \tag{9.1.3}$$

$$\boldsymbol{B_e}(\boldsymbol{x}, s) \to \boldsymbol{B'_e}(\boldsymbol{x}, s) = \boldsymbol{B_e}(\boldsymbol{x}, s - \boldsymbol{b}\cdot\boldsymbol{x}) - \boldsymbol{b} \times \boldsymbol{E_e}(\boldsymbol{x}, s - \boldsymbol{b}\cdot\boldsymbol{x}). \tag{9.1.4}$$

When the limit $\gamma \to 0$ is taken, the linear theory is recovered[1]. Therefore the symmetries of the electric Carrollian limit of ModMax are the same as the ones found for the electric Carrollian limit of Maxwell theory.

9.1.2 Magnetic limit

The magnetic limit is obtained from re-scaling $\boldsymbol{E_m} = \boldsymbol{E}$, $\boldsymbol{B_m} = (C/c)\,\boldsymbol{B}$ and $s = (cC)\,t$ in Maxwell's equations and then taking the limit $C \to \infty$.

$$\frac{\partial \boldsymbol{B_m}}{\partial s} = 0 \tag{9.1.5}$$

$$\nabla \cdot \boldsymbol{B_m} = 0 \tag{9.1.6}$$

$$e^{-\gamma}\left(\nabla \times \boldsymbol{B_m} - \frac{\partial \boldsymbol{E_m}}{\partial s}\right) - 2\sinh\gamma \frac{\boldsymbol{B_m} \cdot \dfrac{\partial \boldsymbol{E_m}}{\partial s}}{B_m^2} \boldsymbol{B_m} = 0 \tag{9.1.7}$$

$$e^{-\gamma}\,\nabla \cdot \boldsymbol{E_m} + 2\sinh\gamma\;(\boldsymbol{B_m} \cdot \nabla)\,\frac{\boldsymbol{B_m} \cdot \boldsymbol{E_m}}{B_m^2} = 0. \tag{9.1.8}$$

In contrast with the electric limit, this one has surviving non-linear terms in both equations coming from the Lagrangian. Equation (9.1.7) can be manipulated in such a way as to eliminate its non-linear contribution, while the non-linear term remains in (9.1.8). Indeed, if we take the dot product with the magnetic field $\boldsymbol{B_m}$,

$$e^{-\gamma}\left(\nabla \times \boldsymbol{B_m} - \frac{\partial \boldsymbol{E_m}}{\partial s}\right) \cdot \boldsymbol{B_m} - 2\sinh\gamma \boldsymbol{B_m} \cdot \frac{\partial \boldsymbol{E_m}}{\partial s} = 0 \tag{9.1.9}$$

$$-e^{\gamma} \boldsymbol{B_m} \cdot \frac{\partial \boldsymbol{E_m}}{\partial s} = 0. \tag{9.1.10}$$

Notice that if we combine this with (9.1.5) we arrive at

$$\frac{\partial}{\partial s}\left(\boldsymbol{E_m} \cdot \boldsymbol{B_m}\right) = \frac{\partial P_m}{\partial s} = 0, \tag{9.1.11}$$

[1]Although in this particular case taking the limit seems irrelevant, when coupling the theory to matter the ModMax case will contain a γ-dependent vacuum permittivity and permeability in contrast to its electric Carrollian Maxwell counterpart.

where P_m is the magnetic Carrollian version of the Lorentz invariant P. This means P_m is constant in time. We have found a non-trivial magnetic Carrollian limit of ModMax theory and delve now into the subject of obtaining and analyzing its symmetries. This is eased by noticing it is possible to use the results presented in section 7.1.1.1. Both equation (9.1.7) and (9.1.8) can be rearranged in such a way as to map them into the shape of the magnetic Carrollian limit of Maxwell theory[2]. This is achieved as follows

$$e^{-\gamma}\nabla \times \boldsymbol{B_m} - \frac{\partial}{\partial s}\left(e^{-\gamma}\boldsymbol{E_m} + 2\sinh\gamma\frac{\boldsymbol{B_m}\cdot\boldsymbol{E_m}}{B_m^2}\boldsymbol{B_m}\right) = 0 \tag{9.1.12}$$

$$\nabla \cdot \left(e^{-\gamma}\boldsymbol{E_m} + 2\sinh\gamma\frac{\boldsymbol{B_m}\cdot\boldsymbol{E_m}}{B_m^2}\boldsymbol{B_m}\right) = 0. \tag{9.1.13}$$

Notice all non-linear contributions are acting as a modification to the electric field $\boldsymbol{E_m}$ and only appear in the pair of equations coming from the Lagrangian, as there is no dependence on the electric field in the remaining pair. Therefore we define

$$\mathfrak{E} = \boldsymbol{E_m} + 2e^{\gamma}\sinh\gamma\frac{\boldsymbol{B_m}\cdot\boldsymbol{E_m}}{B_m^2}\boldsymbol{B_m} \qquad \mathfrak{B} = \boldsymbol{B_m}, \tag{9.1.14}$$

which is invertible, with inverse given by

$$\boldsymbol{E_m} = \mathfrak{E} - 2e^{-\gamma}\sinh\gamma\,\frac{\mathfrak{E}\cdot\mathfrak{B}}{\mathfrak{B}^2}\mathfrak{B} \qquad \boldsymbol{B_m} = \mathfrak{B}. \tag{9.1.15}$$

By performing this transformation, ModMax's Carrollian magnetic limit can be written in the same way as Maxwell's

[2]Remark: while it is true that there exists a bijection between the magnetic Carrollian limit of ModMax theory and that of Maxwell theory, they are not equivalent. This is because there are γ-dependent solutions to the magnetic Carrollian limit of ModMax

$$\frac{\partial \mathfrak{B}}{\partial s} = 0 \qquad \nabla \cdot \mathfrak{B} = 0 \tag{9.1.16}$$

$$\nabla \times \mathfrak{B} - \frac{\partial \mathfrak{E}}{\partial s} = 0 \qquad \nabla \cdot \mathfrak{E} = 0. \tag{9.1.17}$$

The symmetries of these equations were obtained in a previous chapter and can be used to deduce how the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ transform by using the transformations for $\mathfrak{E}$ and $\mathfrak{B}$. We shall start from the transformations that are most difficult to construct, this is

$$h^{\mathcal{P}} : \mathbb{R}^3 \times \mathcal{E}_m \longrightarrow \mathcal{E}_m \tag{9.1.18}$$

$$(\boldsymbol{\lambda}, s, \mathbf{x}, \mathfrak{E}, \mathfrak{B}) \longrightarrow h^{\mathcal{P}} (\boldsymbol{\lambda}, s, \mathbf{x}, \mathfrak{E}, \mathfrak{B}) := (s, \mathbf{x} + \boldsymbol{\lambda}, \mathfrak{E}, \mathfrak{B}) \,. \tag{9.1.19}$$

We have $\boldsymbol{B'} = \mathfrak{B}' = \mathfrak{B} = \boldsymbol{B}$ and the same can be done for the electric field since neither of them transforms. And the second most difficult one, time translations

$$h^{\mathcal{H}} : \mathbb{R} \times \mathcal{E}_m \longrightarrow \mathcal{E}_m \tag{9.1.20}$$

$$(\lambda, s, \mathbf{x}, \mathfrak{E}, \mathfrak{B}) \longrightarrow h^{\mathcal{H}} (\lambda, s, \mathbf{x}, \mathfrak{E}, \mathfrak{B}) := (s + \lambda, \mathbf{x}, \mathfrak{E}, \mathfrak{B}) \,. \tag{9.1.21}$$

Just by the same logic as in the previous case, we have $\boldsymbol{E'} = \boldsymbol{E}$ and $\boldsymbol{B'} = \boldsymbol{B}$.

For rotations it is convenient to consider the general transformation

$$h^{\mathcal{J}} : SO(3) \times \mathcal{E}_m \longrightarrow \mathcal{E}_m \tag{9.1.22}$$

$$(R, s, \mathbf{x}, \mathfrak{E}, \mathfrak{B}) \longrightarrow h^{\mathcal{J}} (R, s, \mathbf{x}, \mathfrak{E}, \mathfrak{B}) := (s, R\mathbf{x}, R\mathfrak{E}, R\mathfrak{B}) \,. \tag{9.1.23}$$

This transformation is used to derive how the electric and magnetic field transform under rotations in the Carrollian magnetic limit of ModMax[3]

[3]Not surprisingly, in the same way as in Maxwell or all the other cases.

$$\boldsymbol{E'} = \mathfrak{E}' - 2e^{-\gamma}\sinh\gamma\frac{\mathfrak{E}'\cdot\mathfrak{B}'}{\mathfrak{B}'\cdot\mathfrak{B}'}\mathfrak{B}' \qquad \boldsymbol{B'} = \mathfrak{B}' \tag{9.1.24}$$

$$= R\mathfrak{E} - 2e^{-\gamma}\sinh\gamma\frac{(R\mathfrak{E})^T R\mathfrak{B}}{(R\mathfrak{B})^T R\mathfrak{B}}R\mathfrak{B} \qquad = R\mathfrak{B} \tag{9.1.25}$$

$$= R\left(\mathfrak{E} - 2e^{-\gamma}\sinh\gamma\,\frac{\mathfrak{E}\cdot\mathfrak{B}}{\mathfrak{B}^2}\mathfrak{B}\right) \qquad = R\boldsymbol{B} \tag{9.1.26}$$

$$= R\boldsymbol{E}. \tag{9.1.27}$$

Super translations are an action of the $\left(C^\infty\left(\mathbb{R}^3\right),+\right)$ additive group with action given by

$$\mathfrak{h}^{\mathcal{M}} : C^\infty\left(\mathbb{R}^3\right)\times\mathcal{E}_m \longrightarrow \mathcal{E}_m \tag{9.1.28}$$

$$(f,s,\mathbf{x},\mathfrak{E},\mathfrak{B}) \longrightarrow h^{\mathcal{M}}(f,s,\mathbf{x},\mathfrak{E},\mathfrak{B}) := (s+f(x,y,z),\mathbf{x},\mathfrak{E}-\nabla f\times\mathfrak{B},\mathfrak{B})\,. \tag{9.1.29}$$

We have already proven that the magnetic field not transforming implies the magnetic field not transforming. Curiously enough, the electric field transforms in the same way as in the Carrollian magnetic limit of Maxwell theory

$$\boldsymbol{E'} = \mathfrak{E}' - 2e^{-\gamma}\sinh\gamma\,\frac{\mathfrak{E}'\cdot\mathfrak{B}'}{\mathfrak{B}'^2}\mathfrak{B}' \tag{9.1.30}$$

$$= \mathfrak{E} - \nabla f\times\mathfrak{B} - 2e^{-\gamma}\sinh\gamma\frac{(\mathfrak{E}-\nabla f\times\mathfrak{B})\cdot\mathfrak{B}}{\mathfrak{B}^2}\mathfrak{B} \tag{9.1.31}$$

$$= \mathfrak{E} - 2e^{-\gamma}\sinh\gamma\frac{\mathfrak{E}\cdot\mathfrak{B}}{\mathfrak{B}^2}\mathfrak{B} - \nabla f\times\mathfrak{B} \tag{9.1.32}$$

$$= \boldsymbol{E} - \nabla f\times\boldsymbol{B}. \tag{9.1.33}$$

Space dilations correspond to the action of the multiplicative group $(\mathbb{R}^\times,\cdot)$, where $\mathbb{R}^\times = \mathbb{R}/\{0\}$

$$h^{\mathcal{D}} : \mathbb{R} \times \mathcal{E}_m \longrightarrow \mathcal{E}_m \tag{9.1.34}$$

$$(\lambda, s, \mathbf{x}, \mathfrak{E}, \mathfrak{B}) \longrightarrow h^{\mathcal{D}}(\lambda, s, \mathbf{x}, \mathfrak{E}, \mathfrak{B}) := \left(s, e^{\lambda}\mathbf{x}, \mathfrak{E}, e^{\lambda}\mathfrak{B}\right). \tag{9.1.35}$$

They have the same transformation rule

$$\boldsymbol{E}' = \mathfrak{E}' - 2e^{-\gamma}\sinh\gamma\frac{\mathfrak{E}'\cdot\mathfrak{B}'}{\mathfrak{B}'^2} \qquad \boldsymbol{B}' = \mathfrak{B}' \tag{9.1.36}$$

$$= \mathfrak{E} - 2e^{-\gamma}\sinh\gamma\frac{e^{2\lambda}\mathfrak{E}\cdot\mathfrak{B}}{e^{2\lambda}\mathfrak{B}^2} \qquad = e^{\lambda}\mathfrak{B} \tag{9.1.37}$$

$$= \boldsymbol{E} \qquad = e^{\lambda}\boldsymbol{B}. \tag{9.1.38}$$

Time dilations correspond to the action of the multiplicative group $(\mathbb{R}^{\times}, \cdot)$, with action given by

$$h^{\mathcal{Q}} : \mathbb{R} \times \mathcal{E}_m \longrightarrow \mathcal{E}_m \tag{9.1.39}$$

$$(\lambda, s, \mathbf{x}, \mathfrak{E}, \mathfrak{B}) \longrightarrow h^{\mathcal{Q}}(\lambda, s, \mathbf{x}, \mathfrak{E}, \mathfrak{B}) := \left(e^{\lambda}s, \mathbf{x}, \mathfrak{E}, e^{-\lambda}\mathfrak{B}\right). \tag{9.1.40}$$

Time translations were expected to behave in the same but opposite way as the spatial ones, which is indeed the case

$$\boldsymbol{E}' = \mathfrak{E}' - 2e^{-\gamma}\sinh\gamma\frac{\mathfrak{E}'\cdot\mathfrak{B}'}{\mathfrak{B}'^2} \qquad \boldsymbol{B}' = \mathfrak{B}' \tag{9.1.41}$$

$$= \mathfrak{E} - 2e^{-\gamma}\sinh\gamma\frac{e^{-2\lambda}\mathfrak{E}\cdot\mathfrak{B}}{e^{-2\lambda}\mathfrak{B}^2} \qquad = e^{-\lambda}\mathfrak{B} \tag{9.1.42}$$

$$= \boldsymbol{E} \qquad = e^{-\lambda}\boldsymbol{B}. \tag{9.1.43}$$

Field dilations are also an action of the multiplicative group $(\mathbb{R}^{\times}, \cdot)$ given by

$$h^{\mathcal{W}} : \mathbb{R} \times \mathcal{E}_m \longrightarrow \mathcal{E}_m \tag{9.1.44}$$
$$(\lambda, s, \mathbf{x}, \mathfrak{E}, \mathfrak{B}) \longrightarrow h^{\mathcal{W}}(\lambda, s, \mathbf{x}, \mathfrak{E}, \mathfrak{B}) := \left(s, \mathbf{x}, e^{\lambda}\mathfrak{E}, e^{\lambda}\mathfrak{B}\right), \tag{9.1.45}$$

and, of course, it implies the same transformation rule in the Carrollian magnetic limit of ModMax

$$\boldsymbol{E'} = \mathfrak{E}' - 2e^{-\gamma}\sinh\gamma\frac{\mathfrak{E}'\cdot\mathfrak{B}'}{\mathfrak{B}'^2}\mathfrak{B}' \tag{9.1.46}$$
$$= e^{\lambda}\mathfrak{E} - 2\sinh\gamma\frac{e^{2\lambda}\mathfrak{E}\cdot\mathfrak{B}}{e^{2\lambda}\mathfrak{B}^2}e^{\lambda}\mathfrak{B} \tag{9.1.47}$$
$$= e^{\lambda}\left(\mathfrak{E} - 2e^{-\gamma}\sinh\gamma\,\frac{\mathfrak{E}\cdot\mathfrak{B}}{\mathfrak{B}^2}\mathfrak{B}\right) \tag{9.1.48}$$
$$= e^{\lambda}\boldsymbol{E}. \tag{9.1.49}$$

This one is an action of the additive group $(\mathbb{R}, +)$

$$h^{\mathcal{U}} : \mathbb{R} \times \mathcal{E}_m \longrightarrow \mathcal{E}_m \tag{9.1.50}$$
$$(\lambda, s, \mathbf{x}, \mathfrak{E}, \mathfrak{B}) \longrightarrow h^{\mathcal{U}}(\lambda, s, \mathbf{x}, \mathfrak{E}, \mathfrak{B}) := (s, \mathbf{x}, \mathfrak{E} - \lambda\mathfrak{B}, \mathfrak{B}). \tag{9.1.51}$$

It may appear at first that the transformation rule for this case is different from its Maxwell counterpart. However, since $e^{-2\gamma}$ is a strictly positive number for any $\lambda \in \mathbb{R}$ is a $\tau \in \mathbb{R}$ given by $\lambda e^{-2\gamma}$ that produces the same transformation

$$\boldsymbol{E'} = \mathfrak{E}' - 2e^{-\gamma}\sinh\gamma\,\frac{\mathfrak{E}'\cdot\mathfrak{B}'}{\mathfrak{B}'^2}\mathfrak{B}' \tag{9.1.52}$$
$$= \mathfrak{E} - \lambda\mathfrak{B} - 2e^{-\gamma}\sinh\gamma\frac{\mathfrak{E}\cdot\mathfrak{B}}{\mathfrak{B}^2}\mathfrak{B} + 2\lambda e^{-\gamma}\sinh\gamma\mathfrak{B} \tag{9.1.53}$$
$$= \boldsymbol{E} - \lambda\left(1 - 2e^{-\gamma}\sinh\gamma\right)\boldsymbol{B} \tag{9.1.54}$$
$$= \boldsymbol{E} - \lambda e^{-2\gamma}\boldsymbol{B}. \tag{9.1.55}$$

Special conformal transformations of order two are characterized by

$$h^{\mathcal{S}_A} : \mathbb{R} \times \mathcal{E}_m \longrightarrow \mathcal{E} \tag{9.1.56}$$

$$(\lambda, s, \mathbf{x}, \mathfrak{E}, \mathfrak{B}) \longrightarrow h^{\mathcal{S}_A}(s, \mathbf{x}, \mathfrak{E}, \mathfrak{B}), \tag{9.1.57}$$

with them being explicitly given by

$$h^{\mathcal{S}_1}(\lambda, s, \mathbf{x}, \mathfrak{E}, \mathfrak{B}) = (\omega_x(\lambda)s, \omega_x(\lambda)(x - \lambda \mathbf{x}\cdot\mathbf{x}), \omega_x(\lambda)y, \omega_x(\lambda)z, T_1(\lambda)\mathfrak{E} + O_1(\lambda)\mathfrak{B}, T_1(\lambda)\mathfrak{B}) \tag{9.1.58}$$

$$h^{\mathcal{S}_2}(\lambda, s, \mathbf{x}, \mathfrak{E}, \mathfrak{B}) = (\omega_y(\lambda)s, \omega_y(\lambda)x, \omega_y(\lambda)(y - \lambda \mathbf{x}\cdot\mathbf{x}), \omega_y(\lambda)z, T_2(\lambda)\mathfrak{E} + O_2(\lambda)\mathfrak{B}, T_2(\lambda)\mathfrak{B}) \tag{9.1.59}$$

$$h^{\mathcal{S}_3}(\lambda, s, \mathbf{x}, \mathfrak{E}, \mathfrak{B}) = (\omega_z(\lambda)s, \omega_z(\lambda)x, \omega_z(\lambda)y, \omega_z(\lambda)(z - \lambda \mathbf{x}\cdot\mathbf{x}), T_3(\lambda)\mathfrak{E} + O_3(\lambda)\mathfrak{B}, T_3(\lambda)\mathfrak{B}). \tag{9.1.60}$$

By using the properties[4] $(O_A(\lambda)\boldsymbol{a}) \cdot (T_A(\lambda)\boldsymbol{b}) = 0$ and $(T_A(\lambda)\boldsymbol{a}) \cdot (T_A(\lambda)\boldsymbol{b}) = \Omega_A(\lambda)\boldsymbol{a}\cdot\boldsymbol{b}$ for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ we can prove the action of them is the same as in the Maxwell case

$$\boldsymbol{E}' = \mathfrak{E}' - 2e^{-\gamma}\sinh\gamma \frac{\mathfrak{E}'\cdot\mathfrak{B}'}{\mathfrak{B}'^2}\mathfrak{B}' \tag{9.1.61}$$

$$= T_A(\lambda)\mathfrak{E} + O_A(\lambda)\mathfrak{B} - 2e^{-\gamma}\sinh\gamma \frac{(T_A(\lambda)\mathfrak{E} + O_A(\lambda)\mathfrak{B})\cdot(T_A(\lambda)\mathfrak{B})}{(T_A(\lambda)\mathfrak{B})\cdot(T_A(\lambda)\mathfrak{B})} T_A(\lambda)\mathfrak{B} \tag{9.1.62}$$

$$= T_A(\lambda)\left(\mathfrak{E} - 2e^{-\gamma}\sinh\gamma \frac{\mathfrak{E}\cdot\mathfrak{B}}{\mathfrak{B}^2}\mathfrak{B}\right) + O_A(\lambda)\mathfrak{B} \tag{9.1.63}$$

$$= T_A(\lambda)\boldsymbol{E} + O_A(\lambda)\boldsymbol{B}. \tag{9.1.64}$$

When the limit $\gamma \to 0$ is taken, the linear theory is recovered. Before concluding this section, let us mention that (9.1.14) and (9.1.15) have to be considered as duality transformations between two different theories, namely magnetic Carrollian

[4] Although it is not relevant to the current proof, each special conformal Carrollian transformation of level two is an additive $(\mathbb{R}, +)$ group action. This means for each $A \in \{1, 2, 3\}$ we have $\mathbb{O}_A(\lambda_1)\mathbb{O}_A(\lambda_2) = \mathbb{O}_A(\lambda_1 + \lambda_2)$ and $\mathbb{T}_A(\lambda_1)\mathbb{T}_A(\lambda_2) = \mathbb{T}_A(\lambda_1 + \lambda_2)$.

ModMax and magnetic Carrollian Maxwell. These transformations prove useful constructing Lie point symmetries of the former but by no means trivialize the Carrollian ModMax theory. Coupling it to matter leads to completely different theories.

9.2 At the level of the Hamiltonian

The Hamiltonian formulation of ModMax was done in the first order formalism in Escobar et al. (2022) in accordance to Plebanski (1970) using the Dirac method described in Dirac (2001). This approach is no use for us, however, as it does not yield solvable momenta and thus cannot be used to take Carrollian limits from it.

Even though we do not have an explicit expression of the ModMax Hamiltonian written in terms of its canonical variables it is still possible for us to arrive at Hamiltonian formulations of both magnetic and electric limits of ModMax by working with our incomplete Hamiltonian formulation of ModMax.

This is done in two equivalent but slightly different ways in what follows. First the Hamiltonian formulations of the limits are built by taking the Carrollian limit of the ModMax momenta and using them to construct the action principles by previous appropriate re-scaling of the electric and magnetic field. Afterwards, the ultrarelativistic limit is taken in the resulting action principle as is done in Henneaux and Salgado-Rebolledo (2021). This two-step limit is unavoidable in the current situation as the use of the Carrollian limits of the canonical momenta is nevertheless needed to arrive at the adequate limits of the equations of motion. The second approach is to consider the ModMax energy function, reparametrized according to the desired limit and then taking the limit, this yields the same result as the previously discussed method and one arrives at the correct equations of motion if the Carrollian limits of the canonical momenta are taken into account.

The first step in the first method is to write the Lagrangian explicitly in terms of the connection in order to be able to find the canonical momenta

$$\mathcal{L} = \cosh\gamma\, S + \sinh\gamma\, \sqrt{S^2 + P^2} \tag{9.2.1}$$

$$= \frac{1}{2}\cosh\gamma\, \left(\frac{E^2}{c^2} - B^2\right) + \sinh\gamma\, \sqrt{\frac{1}{4}\left(\frac{E^2}{c^2} - B^2\right)^2 + \frac{1}{c^2}\left(\boldsymbol{E}\cdot\boldsymbol{B}\right)^2} \tag{9.2.2}$$

$$= \frac{1}{2}\cosh\gamma\, \left(\frac{E^2}{c^2} - B^2\right) + \sinh\gamma\, \sqrt{\frac{1}{4}\left(\frac{1}{c^2}\left(\nabla\phi + \dot{\boldsymbol{A}}\right)^2 - (\nabla\times\boldsymbol{A})^2\right)^2 + \frac{1}{c^2}\left(\left(\nabla\phi + \dot{\boldsymbol{A}}\right)\cdot\nabla\times\boldsymbol{A}\right)^2}. \tag{9.2.3}$$

As expected, there's no dependence on $\dot{\phi}$, which also happened when Maxwell theory was considered and a rigorous study of this Hamiltonian formulation would require an analysis under Dirac's formalism of restrictions, however, that is not needed for the current work.

Therefore $\pi^0 := \dfrac{\partial\mathcal{L}}{\partial\dot{\phi}} = 0$. The only non-zero canonical momenta are

$$\pi^a = \frac{\partial\mathcal{L}}{\partial\dot{A}_a} \tag{9.2.4}$$

$$= \frac{1}{c^2}\cosh\gamma\, \left(\nabla\phi + \dot{\boldsymbol{A}}\right)^a \tag{9.2.5}$$

$$+ \frac{1}{2}\sinh\gamma\, \frac{\frac{1}{c^2}\left[\frac{1}{c^2}\left(\nabla\phi + \dot{\boldsymbol{A}}\right)^2 - (\nabla\times\boldsymbol{A})^2\right]\left(\nabla\phi + \dot{\boldsymbol{A}}\right)^a}{\left(\frac{1}{4}\left[\frac{1}{c^2}\left(\nabla\phi + \dot{\boldsymbol{A}}\right)^2 - (\nabla\times\boldsymbol{A})^2\right]^2 + \frac{1}{c^2}\left[\left(\nabla\phi + \dot{\boldsymbol{A}}\right)\cdot\nabla\times\boldsymbol{A}\right]^2\right)^{1/2}}$$
$$+ \frac{1}{c^2}\sinh\gamma \frac{\left[\left(\nabla\phi + \dot{\boldsymbol{A}}\right)\cdot\nabla\times\boldsymbol{A}\right](\nabla\times\boldsymbol{A})^a}{\left(\frac{1}{4}\left[\frac{1}{c^2}\left(\nabla\phi + \dot{\boldsymbol{A}}\right)^2 - (\nabla\times\boldsymbol{A})^2\right]^2 + \frac{1}{c^2}\left[\left(\nabla\phi + \dot{\boldsymbol{A}}\right)\cdot\nabla\times\boldsymbol{A}\right]^2\right)^{1/2}} \tag{9.2.6}$$

$$= -\frac{1}{c^2}\cosh\gamma\,\boldsymbol{E} - \frac{1}{2c^2}\sinh\gamma\, \frac{\left[\frac{E^2}{c^2} - B^2\right]\boldsymbol{E} + 2\left[\boldsymbol{E}\cdot\boldsymbol{B}\right]\boldsymbol{B}}{\sqrt{\frac{1}{4}\left(\frac{E^2}{c^2} - B^2\right)^2 + \frac{1}{c^2}\left(\boldsymbol{E}\cdot\boldsymbol{B}\right)^2}}. \tag{9.2.7}$$

It may be a hell to solve for $\dot{\boldsymbol{A}}$. Fortunately, solving for $\dot{\boldsymbol{A}}$ is equivalent to solving for $\boldsymbol{E}$. Suppose there's an inverse function f such that

$$\boldsymbol{E} = f\left(\pi, \boldsymbol{B}\right). \tag{9.2.8}$$

Then $\dot{\boldsymbol{A}} = -f\left(\pi, \boldsymbol{B}\right) + \nabla\phi$. Recall that in the Maxwell case Gauss constraint comes from the appearance of $\nabla\phi$ in the canonical momentum as

$$\pi_M = \frac{1}{c^2}\left(\nabla\phi + \dot{\boldsymbol{A}}\right), \tag{9.2.9}$$

and solving for $\dot{\boldsymbol{A}}$ we get

$$\dot{\boldsymbol{A}} = c^2\pi_M - \nabla\phi. \tag{9.2.10}$$

So $\pi \cdot \dot{\boldsymbol{A}} = c^2\pi_M^2 - \pi\cdot\nabla\phi$ and integrating by parts the second term we get $\phi\nabla\cdot\pi_M$, which is the Gauss constraint considering ϕ as a Lagrange multiplier.

In conclusion, we can get the Gauss constraint for ModMax in the same fashion as in Maxwell.

9.2.1 Construction of solvable momenta

Solving for $\boldsymbol{E}$ in equation (9.2.7) is no easy task. An approach for doing so is taking the dot product with the magnetic field $\boldsymbol{B}$, this yields a quartic equation for $\boldsymbol{B}\cdot\boldsymbol{E}$. Replacing the result in (9.2.7) we managed to reduce the dependence on $\boldsymbol{E}$ but not to fully solve the equation.

A more manageable approach to constructing Hamiltonians for the electric and magnetic limits of ModMax is to eliminate the terms of the definition of the momenta which will not contribute to the limit. This is done by means of the introduction of a dimensionless parameter Λ in the same fashion we used the Carrollian velocity C for taking the limits.

9.2.1.1 Electric case

The introduction of the parameter Λ for the electric case is done by field reparametrization as follows

$$\boldsymbol{E} = \boldsymbol{E'} \qquad\qquad \boldsymbol{B} = \frac{1}{\Lambda}\boldsymbol{B'}. \tag{9.2.11}$$

Using this, the canonical momentum becomes

$$\pi_{\text{👻}} = -\frac{1}{c^2}\cosh\gamma\,\boldsymbol{E'} - \frac{1}{2c^2}\sinh\gamma\,\frac{\left(\frac{E'^2}{c^2} - \frac{B'^2}{\Lambda^2}\right)\boldsymbol{E'} + \frac{2}{\Lambda^2}\left(\boldsymbol{E'}\cdot\boldsymbol{B'}\right)\boldsymbol{B'}}{\sqrt{\frac{1}{4}\left(E'^2 - \frac{B'^2}{c^2\Lambda^2}\right)^2 + \frac{1}{c^2\Lambda^2}\left(\boldsymbol{E'}\cdot\boldsymbol{B'}\right)^2}}. \tag{9.2.12}$$

We take the limit $\Lambda \to \infty$ to get the electrical momentum

$$\pi_e = \lim_{\Lambda\to\infty}\pi_{\text{👻}} \tag{9.2.13}$$

$$= -\frac{1}{c^2}e^{\gamma}\boldsymbol{E'} = -\frac{1}{c^2}e^{\gamma}\boldsymbol{E}. \tag{9.2.14}$$

Solving for $\dot{\boldsymbol{A}}$ we get

$$\dot{\boldsymbol{A}} = c^2 e^{-\gamma}\pi_e - \nabla\phi. \tag{9.2.15}$$

Now we give the same treatment to the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\cosh\gamma\left(\frac{E^2}{c^2} - B^2\right) + \sinh\gamma\sqrt{\frac{1}{4}\left(\frac{E^2}{c^2} - B^2\right)^2 + \frac{1}{c^2}\left(\boldsymbol{E}\cdot\boldsymbol{B}\right)^2} \tag{9.2.16}$$

$$= \frac{1}{2}\cosh\gamma\left(\frac{E^2}{c^2} - \frac{B^2}{\Lambda^2}\right) + \sinh\gamma\sqrt{\frac{1}{4}\left(\frac{E^2}{c^2} - \frac{B^2}{\Lambda^2}\right)^2 + \frac{1}{c^2\Lambda^2}\left(\boldsymbol{E}\cdot\boldsymbol{B}\right)}. \tag{9.2.17}$$

And we take the limit $\Lambda \to \infty$

$$\lim_{\Lambda\to\infty} L = \frac{1}{2}e^{\gamma}E^2. \tag{9.2.18}$$

With this we construct the Hamiltonian

$$H^E = \int_\Omega \left[\pi_e \cdot \dot{\boldsymbol{A}} - \mathcal{L}\right] d^3\mathbf{x} \tag{9.2.19}$$

$$= \int_\Omega \left[\frac{1}{2}e^{-\gamma}\pi_e^2 - \pi_e \cdot \nabla\phi\right] d^3\mathbf{x}. \tag{9.2.20}$$

The Hamiltonian obtained via this procedure coincides with the one constructed in Henneaux and Salgado-Rebolledo (2021) for the electric limit of Maxwell theory. Therefore, the electric limit of ModMax is equivalent to Maxwell's.

9.2.1.2 Magnetic case

For the magnetic limit we re-scale the fields following

$$\boldsymbol{E} = \frac{1}{\Lambda}\boldsymbol{E'} \qquad\qquad \boldsymbol{B} = \boldsymbol{B'}, \tag{9.2.21}$$

leading to the expression

$$\pi_{\smiley} = -\frac{1}{c^2\Lambda}\cosh\gamma\,\boldsymbol{E'} - \frac{1}{2c^2\Lambda}\sinh\gamma\,\frac{\left(\frac{E'^2}{c^2\Lambda^2} - B'^2\right)\boldsymbol{E'} + 2\left(\boldsymbol{E'}\cdot\boldsymbol{B'}\right)\boldsymbol{B'}}{\sqrt{\frac{1}{4}\left(\frac{E'^2}{c^2\Lambda^2} - B'^2\right)^2 + \frac{1}{c^2\Lambda^2}\left(\boldsymbol{E'}\cdot\boldsymbol{B'}\right)^2}}. \tag{9.2.22}$$

We wish to preserve only the highest order terms in this expression, which corresponds to $\mathcal{O}(\Lambda^{-1})$. To achieve this, we need to take the following limit

$$\pi_m = \lim_{\Lambda\to\infty}\Lambda\pi_{\smiley} \tag{9.2.23}$$

$$= -\frac{1}{c^2}e^{-\gamma}\boldsymbol{E} - \frac{2}{c^2}\sinh\gamma\,\frac{\boldsymbol{E}\cdot\boldsymbol{B}}{B^2}\boldsymbol{B}. \tag{9.2.24}$$

The tilde was dropped because it became irrelevant at this point. Here we can solve for $\boldsymbol{E}\cdot\boldsymbol{B}$ by taking the dot product of equation (9.2.24) with the magnetic field $\boldsymbol{B}$, which yields

$$\boldsymbol{E}\cdot\boldsymbol{B} = -c^2e^{-\gamma}\pi_m\cdot\boldsymbol{B}. \tag{9.2.25}$$

Allowing us to solve for $\boldsymbol{E}$, and therefore for $\dot{\boldsymbol{A}}$

$$-\boldsymbol{E} = c^2e^{\gamma}\left[\pi_m - 2\sinh\gamma\frac{\pi_m\cdot\boldsymbol{B}}{B^2}\boldsymbol{B}\right] \tag{9.2.26}$$

$$\dot{\boldsymbol{A}} = c^2e^{\gamma}\left[\pi_m - 2\sinh\gamma\frac{\pi_m\cdot\boldsymbol{B}}{B^2}\boldsymbol{B}\right] - \nabla\phi. \tag{9.2.27}$$

The Hamiltonian we get thanks to this is

$$H^M = \int_\Omega \left[\pi_m \cdot \dot{\boldsymbol{A}} - \mathcal{L}\right] d^3\mathbf{x} \tag{9.2.28}$$

$$= \int_\Omega \left[c^2 e^\gamma \pi_m^2 - 2c^2 e^\gamma \sinh\gamma \frac{(\pi_m \cdot \boldsymbol{B})^2}{B^2} - \pi_m \cdot \nabla\phi\right] d^3\mathbf{x} - L \tag{9.2.29}$$

$$= \int_\Omega \left[c^2 e^\gamma \pi_m^2 - 2c^2 e^\gamma \sinh\gamma \frac{(\pi_m \cdot \boldsymbol{B})^2}{B^2} + \phi\, \nabla \cdot \pi_m\right] d^3\mathbf{x} - L, \tag{9.2.30}$$

where L is the ModMax Lagrangian written in terms of the canonical variables.

At this point not only is it convenient but it also is necessary to give the Lagrangian density function the same treatment we've already given the momenta

$$L = \frac{1}{2}\cosh\gamma\, \left(\frac{E^2}{c^2} - B^2\right) + \sinh\gamma \sqrt{\frac{1}{4}\left(\frac{E^2}{c^2} - B^2\right)^2 + \frac{1}{c^2}\left(\boldsymbol{E}\cdot\boldsymbol{B}\right)^2} \tag{9.2.31}$$

$$= \frac{1}{2}\cosh\gamma\, \left(\frac{E^2}{\Lambda^2 c^2} - B^2\right) + \sinh\gamma \sqrt{\frac{1}{4}\left(\frac{E^2}{\Lambda^2 c^2} - B^2\right)^2 + \frac{1}{\Lambda^2 c^2}\left(\boldsymbol{E}\cdot\boldsymbol{B}\right)^2}. \tag{9.2.32}$$

Notice we cannot proceed by simply taking the limit $c \to 0$ in (9.2.30). Taking the limit $\Lambda \to \infty$ in (9.2.32) here we arrive at

$$L = -\frac{1}{2}e^{-\gamma}B^2. \tag{9.2.33}$$

Putting it all back together, we get

$$H^M = \int_\Omega \left[c^2 e^\gamma \pi_m^2 - 2c^2 e^\gamma \sinh\gamma \frac{(\pi_m \cdot \boldsymbol{B})^2}{B^2} + \phi\, \nabla \cdot \pi_m + \frac{1}{2}e^{-\gamma}B^2\right] d^3\mathbf{x}. \tag{9.2.34}$$

Finally, we take the limit $c \to 0$ in accordance to Henneaux and Salgado-Rebolledo (2021)

$$H^M = \int_\Omega \left[\phi\, \nabla \cdot \pi_m + \frac{1}{2} e^{-\gamma} B^2\right] d^3\mathbf{x}. \tag{9.2.35}$$

Two things I would like to remark here, first is this Hamiltonian has the same form as the Maxwell one, second is that the non-linear character comes from the momenta's definition.

The equation of motion for the scalar potential is

$$\dot{\phi} = \frac{\partial \mathcal{H}^M}{\partial \pi^0} - \frac{\partial}{\partial x^i}\left(\frac{\partial \mathcal{H}^M}{\partial\left(\partial_i \pi^0\right)}\right) = 0, \tag{9.2.36}$$

and the equation for it's conjugate momentum is

$$\dot{\pi}^0 = -\frac{\partial \mathcal{H}^M}{\partial \phi} + \frac{\partial}{\partial x^i}\left(\frac{\partial \mathcal{H}^M}{\partial\left(\partial_i \phi\right)}\right) \tag{9.2.37}$$

$$0 = -\nabla \cdot \pi_m \tag{9.2.38}$$

$$= \frac{1}{c^2} e^{-\gamma} \boldsymbol{E} + \frac{2}{c^2} \sinh\gamma\ \left(\boldsymbol{B} \cdot \nabla\right) \frac{\boldsymbol{B} \cdot \boldsymbol{E}}{B^2} \tag{9.2.39}$$

$$= \frac{1}{c^2} e^{-\gamma} \nabla \cdot \boldsymbol{E} + \frac{2}{c^2} \sinh\gamma\ \left[\frac{\left(\boldsymbol{B} \cdot \nabla\right) \boldsymbol{B} \cdot \boldsymbol{E}}{B^2} - \frac{\boldsymbol{B} \cdot \boldsymbol{E}}{B^4}\left(\boldsymbol{B} \cdot \nabla\right) B^2\right]. \tag{9.2.40}$$

There's a pair of equations more to be obtained from this. First we have the equation for the vector potential $\boldsymbol{A}$

$$\dot{A}_a = \frac{\partial \mathcal{H}^M}{\partial \pi^a} - \frac{\partial}{\partial x^i}\left(\frac{\partial \mathcal{H}^M}{\partial\left(\partial_i \pi^a\right)}\right) \tag{9.2.41}$$

$$= -\partial_a \phi. \tag{9.2.42}$$

This is the same kind of inconsistency that appears in the Maxwell case and we must deal with it in the same fashion. But first, the final equation

$$\dot{\pi}^a = -\frac{\partial \mathcal{H}^M}{\partial A_a} + \frac{\partial}{\partial x^i}\left(\frac{\partial \mathcal{H}^M}{\partial\left(\partial_i A_a\right)}\right) \tag{9.2.43}$$

$$-\frac{1}{c^2}\frac{\partial}{\partial t}\left[e^{-\gamma}\boldsymbol{E} + 2\frac{\boldsymbol{E}\cdot\boldsymbol{B}}{B^2}\boldsymbol{B}\right] = \frac{\partial}{\partial x^i}\left(e^{-\gamma}B_k\frac{\partial}{\partial\left(\partial_i A_a\right)}\epsilon^{lmk}\partial_l A_m\right) \tag{9.2.44}$$

$$= \frac{\partial}{\partial x^i}\left(e^{-\gamma}B_k\epsilon^{iak}\right) \tag{9.2.45}$$

$$-\frac{1}{c^2}\frac{\partial}{\partial t}\left[e^{-\gamma}\boldsymbol{E} + 2\sinh\gamma\frac{\boldsymbol{E}\cdot\boldsymbol{B}}{B^2}\boldsymbol{B}\right] = -e^{-\gamma}\left(\nabla\times\boldsymbol{B}\right)^a. \tag{9.2.46}$$

Now, to have a precise match with the equations obtained directly from the EOM in (9.1.5), (9.1.6), (9.1.7) and (9.1.8), we need to reparametrize the electric and magnetic fields as we've done in all of this work

$$\boldsymbol{E} = \frac{c}{C}\boldsymbol{E_m} \qquad\qquad \boldsymbol{B} = \frac{1}{c}\boldsymbol{B_m}. \tag{9.2.47}$$

Recall we also need to consider Carrollian time $s = (cC)$ and the magnetic Carrollian limit of Bianchi identity

$$\frac{\partial \boldsymbol{B_m}}{\partial s} = 0 \qquad\qquad \nabla\cdot\boldsymbol{B_m} = 0. \tag{9.2.48}$$

Putting this all back together we obtain that equation (9.2.46) becomes

$$e^{-\gamma}\left(\nabla\times\boldsymbol{B_m} - \frac{\partial \boldsymbol{E_m}}{\partial s}\right) - 2\sinh\gamma\,\frac{\boldsymbol{B_m}\cdot\dfrac{\partial \boldsymbol{E_m}}{\partial s}}{B_m^2}\boldsymbol{B_m} = 0. \tag{9.2.49}$$

Notice the decision made is consistent with the basic idea that Hamiltonian equations are equivalent to Lagrangian ones for their respective case.

9.2.2 At the level of the Hamiltonian 2: Electric Boogaloo

While the construction shown in the previous subsection consistently leads to the correct equations of motion, the order of limits may raise doubt of its legitimacy. In what follows I explore an alternative method for the derivation of those Hamiltonians.

The main idea is to use equation (8.2.6) to construct the limits directly.

9.2.2.1 Electric case

Recall we constructed the energy density function as

$$H = \int_\Omega \left[\frac{1}{2}\cosh\gamma \left(\frac{E^2}{c^2} + B^2\right) \right.$$
$$\left. + \frac{1}{4}\sinh\gamma \frac{\left(\frac{E^2}{c^2} - B^2\right)\left(\frac{E^2}{c^2} + B^2\right)}{\sqrt{\frac{1}{4}\left(\frac{E^2}{c^2} - B^2\right)^2 + \frac{1}{c^2}\left(\boldsymbol{E}\cdot\boldsymbol{B}\right)^2}} + \phi\nabla\cdot\boldsymbol{\pi} \right] d^3\mathbf{x}. \tag{9.2.50}$$

If this were written in terms of the canonical variables, it would only be needed taking the limit $c \to 0$ in this expression to achieve one of the limits. A possible approach is to switch first to Carrollian units. We consider field reparametrization as follows

$$\boldsymbol{E} = c\boldsymbol{E_e} \qquad \boldsymbol{B} = \frac{1}{cC}\boldsymbol{B_e} \qquad \boldsymbol{\pi} = \frac{1}{c}\boldsymbol{\pi_e} \qquad \phi = c\phi_e. \tag{9.2.51}$$

With this we obtain

$$H = \int_\Omega \left[\frac{1}{2}\cosh\gamma \left(E_e^2 - \frac{B_e^2}{c^2C^2} \right) \right.$$
$$\left. + \frac{1}{4}\sinh\gamma \frac{\left(E_e^2 - \frac{B_e^2}{c^2C^2} \right)\left(E_e^2 + \frac{B_e^2}{c^2C^2} \right)}{\sqrt{\frac{1}{4}\left(E_e^2 - \frac{B_e^2}{c^2C^2} \right)^2 + \frac{1}{c^2C^2}\left(\boldsymbol{E_e} \cdot \boldsymbol{B_e} \right)}} + \phi_e \nabla \cdot \boldsymbol{\pi_e} \right] d^3\mathbf{x}, \quad (9.2.52)$$

and taking the limit $C \to \infty$ here we arrive at

$$H^E = \int_\Omega \left[\frac{1}{2} e^{\gamma} E_e^2 + \phi_e \nabla \cdot \boldsymbol{\pi_e} \right] d^3\mathbf{x}. \quad (9.2.53)$$

Now, to conclude this calculation we must use the definition of the electrical momenta in (9.2.14) and reparametrize it accordingly, this is

$$\boldsymbol{\pi_e} = -e^{\gamma} \boldsymbol{E_e}. \quad (9.2.54)$$

This allows us to write the Hamiltonian in terms of the canonical variables as follows

$$H^E = \int_\Omega \left[\frac{1}{2} e^{-\gamma} \pi_e^2 + \phi_e \nabla \cdot \boldsymbol{\pi_e} \right] d^3\mathbf{x}. \quad (9.2.55)$$

Notice we arrived at the same result as we previously had. This time, though, we have the advantage of being able to call this the electric limit of the ModMax Hamiltonian instead of a Hamiltonian constructed from the limit of the momenta.

9.2.2.2 Magnetic case

For mere convenience, in this case I'll use the "Λ approach" employed in the previous section. The electric and magnetic field are reparametrized as follows

$$\boldsymbol{E} = \frac{1}{\Lambda}\boldsymbol{E'} \qquad \boldsymbol{B} = \boldsymbol{B'}, \tag{9.2.56}$$

replacing in (8.2.6) we arrive at

$$H = \int \left[\frac{1}{2}\cosh\gamma \left(\frac{E^2}{c^2\Lambda^2} + B^2 \right) \right.$$
$$\left. + \frac{1}{4}\sinh\gamma \frac{\left(\frac{E^2}{c^2\Lambda^2} - B^2 \right)\left(\frac{E^2}{c^2\Lambda^2} + B^2 \right)}{\sqrt{\frac{1}{4}\left(\frac{E^2}{c^2\Lambda^2} - B^2 \right)^2 + \frac{1}{c^2\Lambda^2}\left(\boldsymbol{E}\cdot\boldsymbol{B} \right)^2}} + \phi\nabla\cdot\boldsymbol{\pi} \right] d^3\mathbf{x}. \tag{9.2.57}$$

Taking the limit $\Lambda \to \infty$ we arrive at

$$H^M = \int_\Omega \left[\frac{1}{2} e^{-\gamma} B^2 + \phi\nabla\cdot\boldsymbol{\pi_m} \right] \tag{9.2.58}$$

Where π_m is the same as defined in equation (9.2.24). We have therefore arrived as the same formulae as in the previous section.

Chapter 10

Conclusión

En el presente trabajo se encontraron los límites Carrollianos de ModMax (Modified Maxwell), tanto la contracción eléctrica como la magnética. De las cuales solo la contracción magnética posee contribuciones no-lineales no-triviales, puesto que la contracción eléctrica difiere de Maxwell solo por un factor global. Además, se encontró un mapa invertible entre el límite Carrolliano magnético de ModMax y aquel de Maxwell[1] que surge de intercambiar los momentos canónicos asociados al potencial vectorial entre ambas teorías. Pese a esto, se espera que al incluir materia en la formulación se generen diferencias significativas con respecto a sus contrapartes en la teoría de Maxwell al incluir materia en la formulación.

La inclusión de materia constituye una continuación natural a este trabajo y requiere considerar qué sucede con la ecuación de continuidad para la carga eléctrica. Junto a esto, es posible que diferentes formas de acoplar los campos a materia lleve a resultados con dinámicas no triviales, como se ha visto en trabajos en geometrías Carrollianas que incluyen interacciones entre partículas.

El análisis de las simetrías de los límites Carrollianos de Maxwell mediante el método de simetrías de contacto de Lie permitió tanto verificar lo que ya se sabía: que ambos límites son covariantes bajo la acción del grupo conforme Carrolliano de nivel 2, como la obtención de resultados nuevos que complementan esto: la construcción explícita tanto de la acción de las transformaciones conformes especiales Carrollianas de nivel 2 sobre los campos, como de la acción del sector infinito dimensional correspondiente a supertraslaciones en el tiempo

[1] Véase (9.1.14).

Carrolliano sobre los campos; la separación de las dilataciones espacio-temporales en dilataciones espaciales y temporales; el hallazgo de simetrías internas que fueron legadas por los generadores de transformaciones de dualidad. Los generadores encontrados mediante este método fueron exponenciados para construir sus respectivas transformaciones finitas, donde cada una corresponde a un grupo uniparamétrico con parámetro real. Dichas transformaciones, en conjunto con el isomorfismo entre el límite Carrolliano magnético de Maxwell y el límite Carrolliano magnético de ModMax, fueron usadas para mostrar que las simetrías encontradas para este límite de Maxwell corresponden también a aquel de ModMax.

Debido a que el método empleado para encontrar simetrías emplea polinomios cuyo orden hace crecer rápidamente los costos computacionales asociados a los cálculos y a que en todo orden trabajado en el presente escrito se encontraron generadores nuevos correspondientes a las supertraslaciones[2], existe la posibilidad de que existan generadores nuevos a orden superior de los polinomios. Determinar la existencia de estos es también una continuación natural de este trabajo[3].

Formulaciones Hamiltonianas fueron encontradas para ambos límites mediante dos acercamientos sutilmente distintos. El primer método se basa en construir el Hamiltoniano a partir de los límites Carrollianos de los momentos canónicos asociados al potencial vectorial, el segundo método se basa en obtener los respectivos límites Carrollianos de la función de energía para cada caso. Estos acercamientos indirectos fueron consecuencia de la dificultad para resolver las ecuaciones constitutivas (9.2.6) y, notoriamente, reproducen las ecuaciones de movimiento correctas. En esta línea, la construcción de una formulación simpléctica tanto para el límite eléctrico como para el magnético puede ser interesante.

Finalmente, cabe destacar que hemos construido los límites Galileanos de ModMax y obtenido y analizado sus simetrías. Su no inclusión en este trabajo obedece a tres cosas: este escrito es ya suficientemente largo, no encontramos transformaciones conformes especiales y queremos saber por qué y no encontramos contribuciones no-lineales no-triviales en estos límites. Sin embargo, cabe destacar que estas teorías poseen también una separación del generador de dilataciones espacio-temporales en dos y, en ambos casos, estas simetrías poseen acciones no triviales sobre los campos.

[2] Y por consiguiente el proceso nunca truncó.

[3] Encontrar las soluciones a la ecuación (7.1.188) puede servir para esto.

Chapter 11

Conclusion

In the present work, the Carrollian limits of ModMax (Modified Maxwell) were found, both in its electric and magnetic contractions. Of which, only the magnetic contraction possesses non-trivial, non-linear contributions. This is because the electric contraction of ModMax differs from that of Maxwell only in an overall factor. Furthermore, we found an invertible map[1] between the magnetic Carrollian limit of ModMax and its counterpart in Maxwell theory that comes from interchanging the canonical momenta associated with the vector potential between each theory. Nevertheless, it is expected that the inclusion of matter in the formulation generates a significant difference with Carrollian Maxwell theory.

The inclusion of matter constitutes a natural continuation of this work and requires considering what happens to the electric charge continuity equation. Besides, there is a chance that different matter couplings yield to results with non-trivial dynamics, as has been seen in previous works with interacting particles in Carrollian geometries.

The symmetry analysis of the Carrollian limits of Maxwell theory, carried out via Lie point symmetry method, allowed us to both verify something already known: both limits are covariant under the action of the conformal Carrollian group of level 2, as well as some new results that complement this: the explicit construction of both the action of the special conformal Carrollian transformations of level 2 over the fields and that of the infinite dimensional sector of this group, corresponding to super-translations of Carrollian time; the separation of space-

[1] See (9.1.14)

time dilations into space and time dilations; the internal symmetry bequeathed by the duality symmetry generator. Each generator found via this method were exponentiated in order to construct their corresponding finite transformations. Said transformations, among the isomorphism between the magnetic Carrollian limit of ModMax and the magnetic Carrollian limit of Maxwell, were used to show that the symmetries found for Maxwell's case are also symmetries in their ModMax counterpart.

Since the method employed for finding the symmetries relies in polynomials whose order rapidly increases the computational demands for the necessary computations and that for every order used in this work new generators of super-translations were found[2], there exists the possibility that there exists new generators at higher order of the polynomials. Determining whether they exist is also a natural continuation of this work.

Hamiltonian formulations were found for both Carrollian limits by two slightly different approaches. The first approach is based in constructing the Hamiltonian by using the Carrollian limits of the canonical momenta associated to the vector potential. The second method is based in taking the Carrollian limits of the energy function of the theory for each case. These approaches came as a consequence of the difficulty of solving the constitutive equation (9.2.6) and, notoriously enough, yield the correct equations of motion. In this line, the construction of of a symplectic formulation of both the electric and magnetic limit would be an interesting continuation.

Finally, it is worth noting that we have also constructed the Galilean limits of ModMax, obtained their symmetries and analyzed them. This is not included in this work mainly for three reasons: this work is already quite long for a masters thesis, we did not find Galilean special conformal transformations and we want to know why, and we did not find non-trivial, non-linear contributions in these limits. Nevertheless, it must be said that these theories also possess a separation of space-time dilations into space dilations and time dilations, each having non-trivial actions over the fields.

[2]Implying the process never truncated.

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Appendix A

Lie Point Symmetries

The work presented in Cantwell (2002) is summarized in the following.

A1 Lie point symmetries in one dimension

The Lie point symmetries method presented in Chapter eight of Cantwell (2002) deals with finding the symmetries of differential equations written in the form

$$\Phi\left[x, y, y_x, \ldots\right] = 0. \tag{A1.1}$$

Where x denotes an independent variable, y denotes the dependent variable and y_x corresponds to the derivative of y with respect to x.

When dealing with the symmetry groups of differential equations it is needed to determine how they act on both dependent and independent variables. The following shows how to obtain the appropriate transformations for derivatives.

A1.1 Finite construction

We consider the action of a group on the independent variable x and the dependent variable y characterized by a parameter s

$$\tilde{x} = F[x, y, s] \tag{A1.2}$$
$$\tilde{y} = G[x, y, s], \tag{A1.3}$$

such that when $s = 0$ they remain unchanged

$$x = F[x, y, 0] \tag{A1.4}$$
$$y = G[x, y, 0]. \tag{A1.5}$$

From this, we wish to construct how the derivative y_x transform under the group in a manner consistent with (A1.2) and (A1.3). For this, use of the following conditions is used

$$dy - y_x dx = 0 \tag{A1.6}$$
$$d\tilde{y} - \tilde{y}_{\tilde{x}} d\tilde{x} = 0, \tag{A1.7}$$

which is equivalent to asking $y_x = \dfrac{dy}{dx}$ and $\tilde{y}_{\tilde{x}} = \dfrac{d\tilde{y}}{d\tilde{x}}$. We proceed by differentiating equations (A1.2) and (A1.3) to construct said derivative

$$d\tilde{y} = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial y} dy \tag{A1.8}$$
$$d\tilde{x} = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy. \tag{A1.9}$$

This is replaced in (A1.7), where we solve for $\tilde{y}_{\tilde{x}}$.

$$\tilde{y}_{\tilde{x}} = \frac{d\tilde{y}}{d\tilde{x}} = \frac{\frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy}{\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy} = \frac{\frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}\frac{dy}{dx}}{\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}} \tag{A1.10}$$

$$= \frac{G_x + y_x G_y}{F_x + y_x F_y}. \tag{A1.11}$$

Here, both numerator and denominator were divided by dx to give them the form of total derivatives with respect to x. To simplify notation, the operator D is defined as this total derivative

$$D(\cdot) := \frac{d}{dx}(\cdot) = \frac{\partial}{\partial x}(\cdot) + y_x \frac{\partial}{\partial y}(\cdot). \tag{A1.12}$$

This way, equation (A1.11) can be simply expressed as

$$\tilde{y}_{\tilde{x}} = G_{\{1\}}[x, y, y_x, s] := DG\,(DF)^{-1}. \tag{A1.13}$$

This describes the transformation y_x takes as induced from the transformations for x and y. In the same fashion, the transformation for y_{xx} is constructed. We start from the contact conditions

$$dy_x - y_{xx}dx = 0 \tag{A1.14}$$

$$d\tilde{y}_{\tilde{x}} - \tilde{y}_{\tilde{x}\tilde{x}}d\tilde{x} = 0, \tag{A1.15}$$

which, as stated previously, stem from

$$y_{xx} = \frac{dy_x}{dx} \qquad \tilde{y}_{\tilde{x}\tilde{x}} = \frac{d\tilde{y}_{\tilde{x}}}{d\tilde{x}}. \tag{A1.16}$$

The differential of the numerator in this expression can be obtained by differentiating equation (A1.13)

$$d\tilde{y}_{\tilde{x}} = \frac{\partial G_{\{1\}}}{\partial x}dx + \frac{\partial G_{\{1\}}}{\partial y}dy + \frac{\partial G_{\{1\}}}{\partial y_x}dy_x \tag{A1.17}$$

$$d\tilde{x} = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy. \tag{A1.18}$$

This is replaced in (A1.16) and we carry on exactly as before

$$\tilde{y}_{\tilde{x}\tilde{x}} = \frac{d\tilde{y}_{\tilde{x}}}{d\tilde{x}} = \frac{\frac{\partial G_{\{1\}}}{\partial x}dx + \frac{\partial G_{\{1\}}}{\partial y}dy + \frac{\partial G_{\{1\}}}{\partial y_x}dy_x}{\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy} \tag{A1.19}$$

$$= \frac{G_{\{1\}x} + y_x G_{\{1\}y} + y_{xx} G_{\{1\}y_x}}{F_x + y_x F_x} \tag{A1.20}$$

Notice this time D has an extra term. This is because $G_{\{1\}}$ has an extra dependence on y_x and it has to be accounted in the total derivative expression

$$D(\cdot) = \frac{\partial}{\partial x}(\cdot) + y_x \frac{\partial}{\partial y}(\cdot) + y_{xx}\frac{\partial}{\partial y_x}(\cdot). \tag{A1.21}$$

We arrive, then, at the induced transformation of the second derivative

$$\tilde{y}_{\tilde{x}\tilde{x}} = G_{\{2\}}[x, y, y_x, y_{xx}, s] := DG_{\{1\}}\left(DF\right)^{-1}. \tag{A1.22}$$

It has to be noted that this process can be carried over as long as we wish but for the purpose of this work, this is a good place to stop.

A1.2 Infinitesimal construction

While the formulas obtained in the previous section work well to determine whether a differential equation is invariant under the action of a certain group, it gives us no way of obtaining said group from scratch. The purpose of the infinitesimal construction is precisely that. We start by writing the transformations up to first order in the group parameter s

$$\tilde{x} = x + s\,\xi[x,y] \tag{A1.23}$$

$$\tilde{y} = y + s\,\eta[x,y], \tag{A1.24}$$

where ξ and η are the first order coefficient of the Taylor expansion of F and G with respect to s, respectively

$$\xi[x,y] = \left.\frac{\partial F}{\partial s}\right|_{s=0} \qquad \eta[x,y] = \left.\frac{\partial G}{\partial s}\right|_{s=0}. \tag{A1.25}$$

The idea is to make use of this expansion to construct a system of partial differential equations in order to solve for the symmetries. The first step in achieving so is building the infinitesimal versions of the transformations for the derivatives. We start with the transformation for the first derivative by replacing (A1.23) and (A1.24) into (A1.13)

$$\tilde{y}_{\tilde{x}} = \frac{DG}{DF} \tag{A1.26}$$

$$= \frac{y_x + sD\eta}{1 + sD\xi} \tag{A1.27}$$

$$\approx y_x + s\left(D\eta - y_x D\xi\right) \tag{A1.28}$$

$$= y_x + s\left(\eta_x + \left(\eta_y - \xi_x\right) y_x - \xi_y y_x^2\right). \tag{A1.29}$$

Therefore, the infinitesimal transformation for the first derivative is given by

$$\tilde{y}_{\tilde{x}} = y_x + s\,\eta_{\{1\}}[x, y, y_x]. \tag{A1.30}$$

The construction of the infinitesimal transformation of the second derivative follows suit in the same fashion[1]

$$\begin{aligned}
\tilde{y}_{\tilde{x}\tilde{x}} &= \frac{DG_{\{1\}}}{DF} && \text{(A1.31)}\\
&= \frac{y_{xx} + sD\eta_{\{1\}}}{1 + sD\xi} && \text{(A1.32)}\\
&\approx y_{xx} + s\left(D\eta_{\{1\}} - y_{xx}D\xi\right) && \text{(A1.33)}\\
&= y_{xx} + s\left(\eta_{xx} + \left(2\eta_{xy} - \xi_{xx}\right)y_x + \left(\eta_{yy} - 2\xi_{xy}\right)y_x^2 - \xi_{xx}y_x^3\right.\\
&\quad \left. + \left(\eta_y - 2\xi_x\right)y_{xx} - 3\eta_y y_x y_{xx}\right) && \text{(A1.34)}\\
&= y_{xx} + s\eta_{\{2\}}[x, y, y_x, y_{xx}]. && \text{(A1.35)}
\end{aligned}$$

A1.3 Example of use: the symmetry group of $y_{xx} = 0$

The how-to procedure of this method is better shown with a practical example. Let us consider the case of the equation

$$\Psi[x, y, y_x, y_{xx}] = y_{xx} = 0. \quad \text{(A1.36)}$$

Invariance of an equation $\Phi[x, y, y_x, y_{xx}] = 0$ under the group's action means

$$\Phi[\tilde{x}, \tilde{y}, \tilde{y}_{\tilde{x}}, \tilde{y}_{\tilde{x}\tilde{x}}] = \Phi[x, y, y_x, y_{xx}]. \quad \text{(A1.37)}$$

For simplicity, we define $\boldsymbol{z} = (x, y, y_x, y_{xx})$ so that $\tilde{\boldsymbol{z}} = (\tilde{x}, \tilde{y}, \tilde{y}_{\tilde{x}}, \tilde{y}_{\tilde{x}\tilde{x}})$. Expanding an equation $\Phi[\tilde{\boldsymbol{z}}]$ in powers of s

$$\begin{aligned}
\Phi[\tilde{\boldsymbol{z}}] &= \Phi[\boldsymbol{z}] + s\left.\frac{\partial\Phi}{\partial s}\right|_{s=0} + \frac{s^2}{2}\left.\frac{\partial\Phi}{\partial s}\right|_{s=0} + \ldots && \text{(A1.38)}\\
&= \Phi[\boldsymbol{z}] + s\left.\frac{\partial z^i}{\partial s}\right|_{s=0}\frac{\partial\Phi}{\partial z^i} + \frac{s^2}{2}\left.\frac{\partial z^i}{\partial s}\right|_{s=0}\frac{\partial}{\partial z^i}\left(\left.\frac{\partial z^j}{\partial s}\right|_{s=0}\frac{\partial\Phi}{\partial z^j}\right) + \ldots && \text{(A1.39)}
\end{aligned}$$

[1]It is noteworthy to mention it's possible to proceed ad infinitum. For our purposes, however, this is as good a place to stop as any.

This becomes (A1.37) if and only if

$$\left.\frac{\partial z^i}{\partial s}\right|_{s=0} \frac{\partial \Phi}{\partial z^i} = 0. \tag{A1.40}$$

Defining the twice-extended vector

$$X_{\{2\}} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \eta_{\{1\}} \frac{\partial}{\partial y_x} + \eta_{\{2\}} \frac{\partial}{\partial y_{xx}}. \tag{A1.41}$$

We can rewrite this condition as $X_{\{2\}}\Phi = 0$. If the differential equation one is dealing with includes dependencies up to p-th derivatives then one must make use of the p-th extended vector $X_{\{p\}}$ to implement the invariance condition

$$X_{\{p\}} := \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \eta_{\{1\}} \frac{\partial}{\partial y_x} + \cdots + \eta_{\{p\}} \frac{\partial}{\partial y_{px}}. \tag{A1.42}$$

Returning to our case of study, equation (A1.36) is invariant if

$$\mathcal{L}_{X_{\{2\}}} \Psi = X_{\{2\}} \Psi = 0. \tag{A1.43}$$

This implies $\eta_{\{2\}} = 0$, which is to be expected as the second derivative should not transform in this case

$$0 = \eta_{\{2\}} \tag{A1.44}$$

$$= \eta_{xx} + (2\eta_{xy} - \xi_{xx})\, y_x + (\eta_{yy} - 2\xi_{xy})\, y_x^2 - \xi_{xx} y_x^3. \tag{A1.45}$$

The functions y, their powers and derivatives are linearly independent. So for this condition to held true it follows all coefficients must be simultaneously zero

$$\eta_{xx} = \tag{A1.46}$$

$$2\eta_{xy} - \xi_{xx} = 0 \tag{A1.47}$$

$$\eta_{yy} - 2\xi_{xy} = 0 \tag{A1.48}$$

$$\xi_{yy} = 0. \tag{A1.49}$$

Solving this trough regular methods is quite simple. However, the systems of equations this method produces are, more often than not, over-determined and polynomial expressions are used for solving them. We have

$$\xi = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 + a_7x^3 + a_8x^2y + a_9xy^2 + a_{10}y^3 \tag{A1.50}$$

$$\eta = b_1 + b_2x + b_3y + b_4x^2 + b_5xy + b_6y^2 + b_7x^3 + b_8x^2y + b_9xy^2 + b_{10}y^3. \tag{A1.51}$$

Replacing this back into the equations we get an eight-parameter solution for them

$$\xi = a_1 + a_2x + a_3y + a_4x^2 + a_5xy \tag{A1.52}$$

$$\eta = b_1 + b_2x + b_3y + a_4xy + a_5y^2. \tag{A1.53}$$

We use them to construct both the infinitesimal version of the transformations that leave invariant the equation $y_{xx} = 0$ by taking all parameters to be zero except one

$$a_1 : \tilde{x} = x + s \qquad \tilde{y} = y \tag{A1.54}$$
$$a_2 : \tilde{x} = x + sx \qquad \tilde{y} = y \tag{A1.55}$$
$$a_3 : \tilde{x} = x + sy \qquad \tilde{y} = y \tag{A1.56}$$
$$a_4 : \tilde{x} = x + sx^2 \qquad \tilde{y} = y + sxy \tag{A1.57}$$
$$a_5 : \tilde{x} = x + sxy \qquad \tilde{y} = y + sy^2 \tag{A1.58}$$
$$b_1 : \tilde{x} = x \qquad \tilde{y} = y + s \tag{A1.59}$$
$$b_2 : \tilde{x} = x \qquad \tilde{y} = y + sx \tag{A1.60}$$
$$b_3 : \tilde{x} = x \qquad \tilde{y} = y + sy. \tag{A1.61}$$

And the vector fields[2] that form the algebra of symmetries of the system are given by

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \tag{A1.62}$$
$$= \left(a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy\right) \frac{\partial}{\partial x} + \left(b_1 + b_2 x + b_3 y + a_4 xy + a_5 y^2\right) \frac{\partial}{\partial y}, \tag{A1.63}$$

this is an eight-dimensional real vector space $\mathcal{A}$, with basis

[2]In their non-extended version.

$$A_1 = \frac{\partial}{\partial x} \tag{A1.64}$$

$$A_2 = x\frac{\partial}{\partial x} \tag{A1.65}$$

$$A_3 = y\frac{\partial}{\partial x} \tag{A1.66}$$

$$A_4 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} \tag{A1.67}$$

$$A_5 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} \tag{A1.68}$$

$$B_1 = \frac{\partial}{\partial y} \tag{A1.69}$$

$$B_2 = x\frac{\partial}{\partial y} \tag{A1.70}$$

$$B_3 = y\frac{\partial}{\partial y}. \tag{A1.71}$$

Their commutator table is built by simply evaluating the commutators while understanding $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ as vectors in the tangent space of $M = (x, y)$.

Each vector in this algebra corresponds to the tangent to a curve $\gamma^X : \mathbb{R} \longrightarrow M$ parameterized by s and we can reconstruct it by solving the differential equation

$$\dot{\gamma}^X(s) = X_{\gamma^X(s)}. \tag{A1.72}$$

Proceeding with the computation of the symmetries of $y_{xx} = 0$ we calculate the solutions of the system of ordinary differential equations for A_1

$$\dot{\gamma}^{A_1}(s) = A_{1\,\gamma^{A_1}(s)} \tag{A1.73}$$

$$\dot{x}(s)\frac{\partial}{\partial x} + \dot{y}(s)\frac{\partial}{\partial y} = \frac{\partial}{\partial x}. \tag{A1.74}$$

Initial conditions $x(0) = x_0$ and $y(0) = y_0$ are imposed into the solution of this equation. Let $\gamma_p^X : \mathbb{R} \longrightarrow M$ be the unique solution to equation (A1.72) with initial conditions $\gamma_p^X(0) = p$. Then we get[3]

$$\gamma^{A_1}_{(x_0,y_0)}(s) = (x_0 + s, y_0)\,. \tag{A1.75}$$

This solution is used to construct the transformation associated to A_1 by the flow

$$h^{A_1} : \mathbb{R} \times M \longrightarrow M \tag{A1.76}$$

$$(s, x, y) \longrightarrow h^{A_1}(s, x, y) := \gamma^{A_1}_{(x,y)}(s). \tag{A1.77}$$

For the symmetry generated by A_2 a system of ODEs is constructed for a curve γ^{A_2} by imposing it have tangent vector $A_{2\,\gamma^{A_2}}$

$$\dot\gamma^{A_2}(s) = A_{2\,\gamma^{A_2}(s)} \tag{A1.78}$$

$$\dot x(s)\frac{\partial}{\partial x} + \dot y(s)\frac{\partial}{\partial y} = x(s)\frac{\partial}{\partial x}. \tag{A1.79}$$

The unique solution to this system of ODEs with initial conditions $\gamma^{A_2}(0) = (x_0, y_0)$ is

$$\gamma^{A_2}_{(x_0,y_0)}(s) = (e^s x_0, y_0)\,. \tag{A1.80}$$

This solution is used to construct the finite transformation for A_2

$$h^{A_2} : \mathbb{R} \times M \longrightarrow M \tag{A1.81}$$

$$(s, x, y) \longrightarrow h^{A_2}(s, x, y) := \gamma^{A_2}_{(x,y)}(s). \tag{A1.82}$$

The system of ODEs to solve for the vector field A_3 is

[3] If notation starts to feel a bit crowded I urge you, dear reader, to bear with me.

$$\dot{\gamma}^{A_3}(s) = A_{3\,\gamma^{A_3}(s)} \tag{A1.83}$$
$$\dot{x}(s)\frac{\partial}{\partial x} + \dot{y}(s)\frac{\partial}{\partial y} = y(s)\frac{\partial}{\partial x}. \tag{A1.84}$$

This has a unique solution with initial conditions $\gamma^{A_3}(0) = (x_0, y_0)$ given by

$$\gamma^{A_3}_{(x_0,y_0)}(s) = (x_0 + y_0 s, y_0)\,. \tag{A1.85}$$

The flow of this system of equations yields the transformation associated with A_3

$$h^{A_3} : \mathbb{R} \times M \longrightarrow M \tag{A1.86}$$
$$(s, x, y) \longrightarrow h^{A_3}(s, x, y) := \gamma^{A_3}_{(x,y)}(s). \tag{A1.87}$$

The system of ODEs to solve for finding the symmetry transformation generated by A_4 is

$$\dot{\gamma}^{A_4}(s) = A_{4\,\gamma^{A_4}(s)} \tag{A1.88}$$
$$\dot{x}(s)\frac{\partial}{\partial x} + \dot{y}(s)\frac{\partial}{\partial y} = x(s)^2\frac{\partial}{\partial x} + x(s)y(s)\frac{\partial}{\partial y}. \tag{A1.89}$$

This has unique solution with initial conditions $\gamma^{A_4}(0) = (x_0, y_0)$ given by

$$\gamma^{A_4}_{(x_0,y_0)}(s) = \left(\frac{x_0}{1 - s x_0}, \frac{y_0}{1 - s x_0}\right). \tag{A1.90}$$

Using this solution the flow that represents the action of this symmetry is constructed as

$$h^{A_4} : \mathbb{R} \times M \longrightarrow M \tag{A1.91}$$
$$(s, x, y) \longrightarrow h^{A_4}(s, x, y) := \gamma^{A_4}_{(x,y)}(s). \tag{A1.92}$$

The system of ordinary differential equations[4] to solve for the vector field A_5 is

$$\dot{\gamma}^{A_5}(s) = A_{5\,\gamma^{A_5}(s)} \tag{A1.93}$$
$$\dot{x}(s)\frac{\partial}{\partial x} + \dot{y}(s)\frac{\partial}{\partial y} = x(s)y(s)\frac{\partial}{\partial x} + y(s)^2\frac{\partial}{\partial y}. \tag{A1.94}$$

The unique solution for this system with initial conditions $\gamma^{A_5}(0) = (x_0, y_0)$ is

$$\gamma^{A_5}_{(x_0,y_0)}(s) = \left(\frac{x_0}{1 - sy_0}, \frac{y_0}{1 - sy_0}\right). \tag{A1.95}$$

This is used to construct the flow

$$h^{A_5} : \mathbb{R} \times M \longrightarrow M \tag{A1.96}$$
$$(s, x, y) \longrightarrow h^{A_5}(s, x, y) := \gamma^{A_5}_{(x,y)}(s). \tag{A1.97}$$

The system of ODEs to solve in the B_1 case are

$$\dot{\gamma}^{B_1}(s) = B_{1\,\gamma^{B_1}(s)} \tag{A1.98}$$
$$\dot{x}(s)\frac{\partial}{\partial x} + \dot{y}(s)\frac{\partial}{\partial y} = \frac{\partial}{\partial y}. \tag{A1.99}$$

The unique solution to this with initial conditions $\gamma^{B_1}(0) = (x_0, y_0)$ is the curve

[4] I am trying (and failing) to write this in slightly different ways. I am sure at this point you get the gist of it anyway.

$$\gamma^{B_1}_{(x_0,y_0)}(s) = (x_0, y_0 + s)\,. \tag{A1.100}$$

Using this to construct the symmetry transformation we were looking for via flow we get

$$h^{B_1} : \mathbb{R} \times M \longrightarrow M \tag{A1.101}$$
$$(s, x, y) \longrightarrow h^{B_1}(s, x, y) := \gamma^{B_1}_{(x,y)}(s). \tag{A1.102}$$

The system of ordinary differential equations that serve as imposition of γ^{B_2} having tangent vector $B_{2\,\gamma^{B_2}}$ is

$$\dot{\gamma}^{B_2}(s) = B_{2\,\gamma^{B_2}(s)} \tag{A1.103}$$
$$\dot{x}(s)\frac{\partial}{\partial x} + \dot{y}(s)\frac{\partial}{\partial y} = x(s)\frac{\partial}{\partial y}. \tag{A1.104}$$

The unique solution to this system with initial conditions $\gamma^{B_2}(0) = (x_0, y_0)$ is

$$\gamma^{B_2}_{(x_0,y_0)}(s) = (x_0, y_0 + sx_0)\,. \tag{A1.105}$$

This is used to build the flow that represents the transformation we were looking for

$$h^{B_2} : \mathbb{R} \times M \longrightarrow M \tag{A1.106}$$
$$(s, x, y) \longrightarrow h^{B_2}(s, x, y) := \gamma^{B_2}_{(x,y)}(s). \tag{A1.107}$$

The system of ODEs to solve in this last case is

$$\dot{\gamma}^{B_3}(s) = B_{3\,\gamma^{B_3}(s)} \tag{A1.108}$$
$$\dot{x}(s)\frac{\partial}{\partial x} + \dot{y}(s)\frac{\partial}{\partial y} = y(s)\frac{\partial}{\partial s}. \tag{A1.109}$$

The unique solution to this equations with initial conditions $\gamma^{B_3}(0) = (x_0, y_0)$ is

$$\gamma^{B_3}_{(x_0,y_0)}(s) = (x_0, e^s y_0)\,. \tag{A1.110}$$

The symmetry transformation for B_3 is constructed thusly

$$h^{B_3} : \mathbb{R} \times M \longrightarrow M \tag{A1.111}$$
$$(s, x, y) \longrightarrow h^{B_3}(s, x, y) := \gamma^{B_3}_{(x,y)}(s). \tag{A1.112}$$

Each flow corresponds to a one-parameter subgroup of the symmetries of the equation $y_{xx} = 0$ and from them the total group can be reconstructed. We start by defining the endomorphisms h^X_s on M for each vector field $X \in \mathcal{S}$

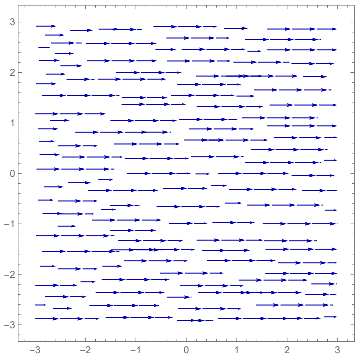
$$h^X_s : M \longrightarrow M \tag{A1.113}$$
$$p \longrightarrow h^X_s(p) := h^X(s, p). \tag{A1.114}$$

Where $s \in \mathbb{R}$ is the parameter of the transformation. The group of symmetries of the equation y_{xx} is the set of transformations

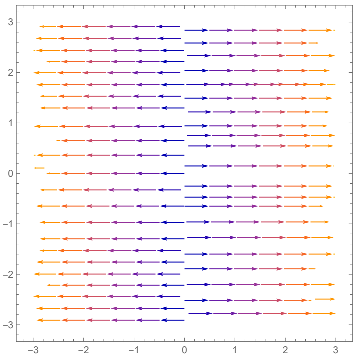
$$T := \left\{ h^X_s \,\middle|\, X \in \mathcal{S} \wedge s \in \mathbb{R} \right\}, \tag{A1.115}$$

with product given by composition of maps.

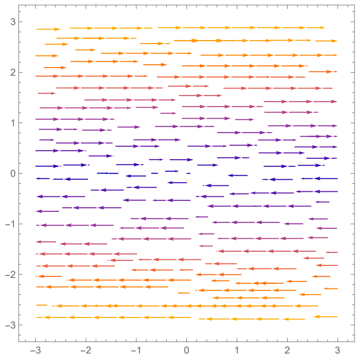
The effect caused by the set of transformations $h_s^X \in T$ for $X \in \mathcal{S}$ and $s \in \mathbb{R}$ can be nicely shown in graphical form in two dimensions by plotting the integral curves of each vector field X. Graphs representing the integral curves of each vector $X \in \mathcal{S}$ are shown below:



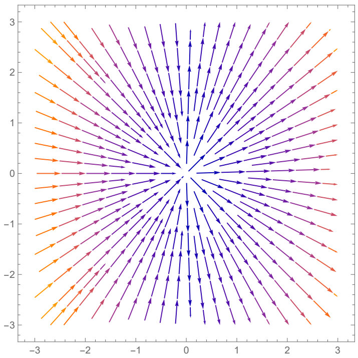
(a) Integral curves of A_1



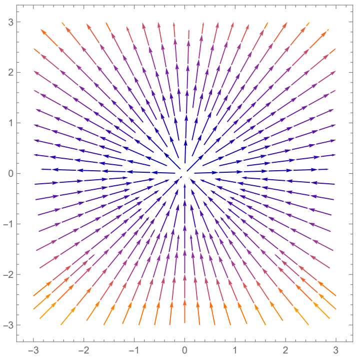
(b) Integral curves of A_2



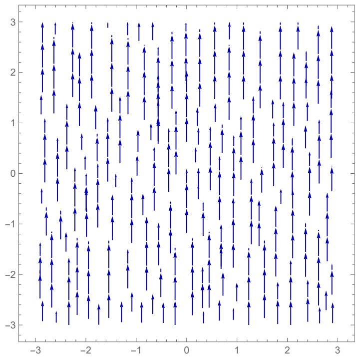
(c) Integral curves of A_3



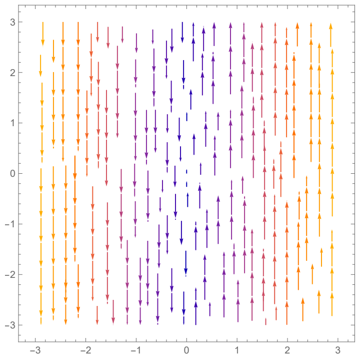
(d) Integral curves of A_4



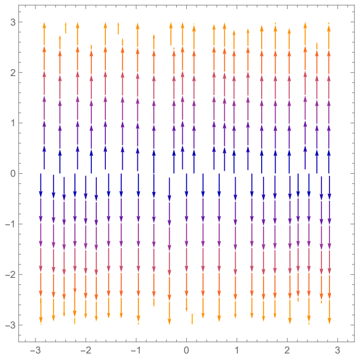
(e) Integral curves of A_5



(f) Integral curves of B_1



(g) Integral curves of B_2



(h) Integral curves of B_3

These were constructed with the software *Mathematica*. Even if this may suggest otherwise, all vector fields are complete. Colour in these graphs gives an idea of "length" of tangent vectors at each point, with warmer colour indicating increasing length and colder ones indicating shorter.

A2 Lie point symmetries in multiple dimensions

Extending this method for multiple dimensions to be able to deal with both partial derivatives and multiple dependent variables is a necessary extension for this work and quite a natural follow-up. PDEs are treated as differential functions in the tangent to a space $(\boldsymbol{x}, \boldsymbol{y})$, where $\boldsymbol{x}$ are independent variables and $\boldsymbol{y}$ are dependent ones. Transformations in this formalism are guaranteed to form a group in the same fashion as the previous case.

We start by considering the finite transformations $F^j[\boldsymbol{x}, \boldsymbol{y}, s]$ for independent variables $\boldsymbol{x}$ and $G^i[\boldsymbol{x}, \boldsymbol{y}, s]$ for dependent variables $\boldsymbol{y}$, with $j \in \{1, ..., n\} \subseteq \mathbb{N}$ and $i \in \{1, ..., m\} \subseteq \mathbb{N}$ being labels for $\boldsymbol{x}$'s and $\boldsymbol{y}$'s coordinates and $s \in \mathbb{R}$ being a real valued parameter characterizing the transformations

$$\tilde{x}^j = F^j[\boldsymbol{x}, \boldsymbol{y}, s] \tag{A2.1}$$

$$\tilde{y}^i = G^i[\boldsymbol{x}, \boldsymbol{y}, s], \tag{A2.2}$$

with $x^j = F^j[\boldsymbol{x}, \boldsymbol{y}, 0]$ and $y^i = G^i[\boldsymbol{x}, \boldsymbol{y}, 0]$. The approach to take in order to arrive at a differential system of equations whose solutions are the coefficients of vector fields generating the symmetries of a given system of PDEs is simply an extension of simpler, previous case. Starting with the contact condition for first order partial derivatives

$$d\tilde{y}^i - \tilde{y}^i_\alpha d\tilde{x}^\alpha = 0. \tag{A2.3}$$

We expand the differentials $d\tilde{x}^j$ and $d\tilde{y}^i$ using (A2.1) and (A2.2), respectively

$$d\tilde{x}^{\alpha} = dF^{j} \qquad\qquad d\tilde{y}^{i} = dG^{i} \tag{A2.4}$$

$$= \frac{\partial F^{j}}{\partial x^{\alpha}} dx^{\alpha} + \frac{\partial F^{j}}{\partial y^{\beta}} dy^{\beta} \qquad\qquad = \frac{\partial G^{i}}{\partial x^{\alpha}} dx^{\alpha} + \frac{\partial G^{i}}{\partial y^{\beta}} dy^{\beta} \tag{A2.5}$$

$$= \left(\frac{\partial}{\partial x^{\alpha}} + y^{\beta}_{\alpha} \frac{\partial}{\partial y^{\beta}} \right) F^{j} dx^{\alpha} \qquad\qquad = \left(\frac{\partial}{\partial x^{\alpha}} + y^{\beta}_{\alpha} \frac{\partial}{\partial y^{\beta}} \right) G^{i} dx^{\alpha} \tag{A2.6}$$

$$= \left(D_{\beta} F^{\alpha} \right) dx^{\beta} \qquad\qquad = \left(D_{\beta} G^{i} \right) dx^{\beta}. \tag{A2.7}$$

Replacing equations (A2.7) into (A2.3) we arrive at

$$\left(D_{\beta} G^{i} - \tilde{y}^{i}_{\alpha} D_{\beta} F^{\alpha} \right) dx^{\beta} = 0. \tag{A2.8}$$

Differentials dx^{α} are linearly independent, therefore each coefficient has to be zero on their own. On the other hand, in order for $D_{\beta} F^{\alpha}$ to have an inverse it needs to satisfy $\det \left(D_{\beta} F^{\alpha} \right) \neq 0$. Imposing this it follows that

$$D_{\beta} G^{i} - \tilde{y}^{i}_{\alpha} D_{\beta} F^{\alpha} = 0 \tag{A2.9}$$

$$\left(D_{\beta} G^{i} - \tilde{y}^{i}_{\alpha} D_{\beta} F^{\alpha} \right) \left(D_{\mu} F^{\beta} \right)^{-1} = 0 \tag{A2.10}$$

$$\tilde{y}^{i}_{\mu} - D_{\beta} G^{i} \left(D_{\mu} F^{\beta} \right)^{-1} = 0. \tag{A2.11}$$

With this we can define the transformations of first derivatives y^{i}_{j} in the once-extended group as

$$\tilde{y}^{i}_{j} = G^{i}_{\{j\}} \left[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y_1}, s \right] := D_{\beta} G^{i} \left(D_{j} F^{\beta} \right)^{-1}, \tag{A2.12}$$

where $\boldsymbol{y_1}$ stands as a shorthand for first derivatives of dependent variables $\boldsymbol{y}$. Next step needed to arrive at a useful point for us is constructing the finite transformations for second derivatives[5].

We start with the contact condition for them

$$d\tilde{y}^i_j - \tilde{y}^i_{j\alpha} dx^\alpha = 0. \tag{A2.13}$$

Replacing (A2.12) into (A2.13) we get

$$\begin{aligned} d\tilde{y}^i_j &= dG^i_{\{j\}} && \text{(A2.14)} \\ &= \frac{\partial G^i_{\{j\}}}{\partial x^\alpha} dx^\alpha + \frac{\partial G^i_{\{j\}}}{\partial y^\beta} dy^\beta + \frac{\partial G^i_{\{j\}}}{\partial y^\beta_\mu} dy^\beta_\mu && \text{(A2.15)} \\ &= \left(\frac{\partial}{\partial x^\alpha} + y^\beta_\alpha \frac{\partial}{\partial y^\beta} + y^\beta_{\mu\alpha} \frac{\partial}{\partial y^\beta_\mu} \right) G^i_{\{j\}} dx^\alpha && \text{(A2.16)} \\ &= D_\alpha G^i_{\{j\}} dx^\alpha. && \text{(A2.17)} \end{aligned}$$

With this it is possible to define finite transformations for second derivatives by following the same steps as before as

$$\tilde{y}^i_{j_1 j_2} = G^i_{\{j_1 j_2\}} [\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y_1}, \boldsymbol{y_2}, s] := D_\alpha G^i_{\{j_1\}} \left(D_{j_2} F^\alpha \right)^{-1}. \tag{A2.18}$$

Now we've got transformation rules for second derivatives we need to construct the differential version of these transformation rules so it is possible to apply them in the problems we are interested in. Consider the infinitesimal version of the transformation for the independent variables $\boldsymbol{x}$ and the dependent variables $\boldsymbol{y}$

$$\begin{aligned} \tilde{x}^j &= x^j + s\xi^j[\boldsymbol{x}, \boldsymbol{y}] && \text{(A2.19)} \\ \tilde{y}^i &= y^i + s\eta^i[\boldsymbol{x}, \boldsymbol{y}], && \text{(A2.20)} \end{aligned}$$

[5]This is because the present work has examples which feature second derivatives. If that were not the case, stopping at first derivatives would have been fine as both Maxwell and ModMax theory are systems of first order partial differential equations.

where $j \in \{1, \ldots, n\} \subseteq \mathbb{N}$ and $i \in \{1, \ldots, m\} \subseteq \mathbb{N}$ and functions ξ^j and η^i are first order terms of Taylor expansions around group parameter s of F^j and G^i, respectively

$$\xi^j[\boldsymbol{x}, \boldsymbol{y}] = \left.\frac{\partial F^j}{\partial s}\right|_{s=0} \qquad \eta^i[\boldsymbol{x}, \boldsymbol{y}] = \left.\frac{\partial G^i}{\partial s}\right|_{s=0}. \tag{A2.21}$$

Replacing (A2.19) and (A2.20) into (A2.12) we get

$$\begin{aligned} \tilde{y}^i_j &= D_\beta \left(y^i + s\eta^i[\boldsymbol{x}, \boldsymbol{y}]\right)^i \left(D_j \left(x^\beta + s\xi^\beta[\boldsymbol{x}, \boldsymbol{y}]\right)\right)^{-1} && \text{(A2.22)} \\ &= \left(y^i_\beta + sD_\beta \eta^i[\boldsymbol{x}, \boldsymbol{y}]\right) \left(\delta^\beta_j + sD_j \xi^\beta[\boldsymbol{x}, \boldsymbol{y}]\right)^{-1} && \text{(A2.23)} \\ &\approx \left(y^i_\beta + sD_\beta \eta^i[\boldsymbol{x}, \boldsymbol{y}]\right) \left(\delta^\beta_j - sD_j \xi^\beta[\boldsymbol{x}, \boldsymbol{y}]\right) && \text{(A2.24)} \\ &= y^i_j + s\left(D_j \eta^i[\boldsymbol{x}, \boldsymbol{y}] - y^i_\beta D_j \xi^\beta[\boldsymbol{x}, \boldsymbol{y}]\right) + \mathcal{O}\left(s^2\right), && \text{(A2.25)} \end{aligned}$$

where the matrix $\left(\delta^{\beta_j} + sD_j\xi^\beta[\boldsymbol{x}, \boldsymbol{y}]\right)^{-1}$ was approximated as[6] $\delta^{\beta_j} - sD_j\xi^\beta[\boldsymbol{x}, \boldsymbol{y}]$ and the last line is the result is truncated at first order in group parameter s. With this we define the infinitesimal transformation of first derivatives

$$\tilde{y}^i_j = y^i_j + s\eta^i_{\{j\}}\left[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y_1}\right], \tag{A2.26}$$

with $\boldsymbol{y_1}$ being a shorthand for first derivatives of dependent variables $\boldsymbol{y}$ and functions $\eta^i_{\{j\}}$ defined as

$$\eta^i_{\{j\}}\left[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y_1}\right] := D_j\eta^i[\boldsymbol{x}, \boldsymbol{y}] - y^i_\beta D_j \xi^\beta[\boldsymbol{x}, \boldsymbol{y}]. \tag{A2.27}$$

Just as in the ODE case, dependence on derivatives of the same order is linear.

For second order partial derivatives the process is repeated, starting by replacing

[6] Which is the usual way of approximating matrices' inverses around a small parameter at the identity.

(A2.19) and (A2.26) into (A2.18)

$$\begin{aligned}
\tilde{y}^i_{j_1 j_2} &= D_\alpha \left(y^i_{j_1} + s\eta^i_{\{j_1\}}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y_1}]\right) \left(D_{j_2}\left(x^\alpha + s\xi^\alpha[\boldsymbol{x}, \boldsymbol{y}]\right)\right)^{-1} && \text{(A2.28)}\\
&= \left(y^i_{j_1\alpha} + sD_\alpha\eta^i_{\{j_1\}}\left[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y_1}\right]\right) \left(\delta^\alpha_{j_2} + sD_{j_2}\xi^\alpha[\boldsymbol{x}, \boldsymbol{y}]\right)^{-1} && \text{(A2.29)}\\
&\approx \left(y^i_{j_1\alpha} + sD_\alpha\eta^i_{\{j_1\}}\left[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y_1}\right]\right) \left(\delta^\alpha_{j_2} - sD_{j_2}\xi^\alpha[\boldsymbol{x}, \boldsymbol{y}]\right) && \text{(A2.30)}\\
&= y^i_{j_1 j_2} + s\left(D_{j_2}\eta^i_{\{j_1\}}\left[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y_1}\right] - y^i_{j_1\alpha}D_{j_2}\xi^\alpha\left[\boldsymbol{x}, \boldsymbol{y}\right]\right) + \mathcal{O}\left(s^2\right), && \text{(A2.31)}
\end{aligned}$$

where the same approximations were taken as the previous case were taken. Care must be put to consider the appropriate version of derivatives D_α to use in each case to arrive at the correct expressions when expanding these transformation rules[7]. Last expression allows us to define the infinitesimal transformation of second derivatives as

$$\tilde{y}^i_{j_1 j_2} = y^i_{j_1 j_2} + s\eta^i_{\{j_1 j_2\}}\left[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y_1}, \boldsymbol{y_2}\right], \tag{A2.32}$$

where $\boldsymbol{y_2}$ stands as a shorthand for second derivatives of dependent variables $\boldsymbol{y}$ and functions $\eta^i_{\{j_1 j_2\}}$ are defined as

$$\eta^i_{\{j_1 j_2\}}\left[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y_1}, \boldsymbol{y_2}\right] := D_{j_2}\eta^i_{\{j_1\}}\left[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y_1}\right] - y^i_{j_1\alpha}D_{j_2}\xi^\alpha\left[\boldsymbol{x}, \boldsymbol{y}\right]. \tag{A2.33}$$

This concludes the mathematical background of the method used here to obtain symmetries[8]. Actual computations were generally done in *Mathematica* software as solving systems of around 400 partial differential equations by hand is a really good way of making dumb mistakes.

[7] Just as when dealing with one variable, derivatives D_β can be thought as representing total derivatives with respect to dependent variable x^β.

[8] If you, dear reader, were looking for an example of the multi-variable version in action I'm happy to disappoint.

Appendix B

Symmetries of the scalar wave equation

This appendix serves as an expansion of chapter 5, where the symmetries of the equations of motion for the relativistic scalar field were not included as to not disturb the narrative. This brief chapter contains the symmetries of the wave equation[1]

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0, \tag{B.0.1}$$

which consists of space-time translations, space rotations, Lorentzian boosts, space-time dilations and special conformal transformations. The symmetries of the wave equation consist, then, of the 4-dimensional relativistic conformal group that was introduced as a natural extension of the Poincaré group ISO $(3,1)$. These symmetries were obtained via the same procedure used in the rest of this work.

[1]In the terminology used in the previous appendix, this equation can be written as $\Phi[t, x, y, z, \phi_{tt}, \phi_{xx}, \phi_{yy}, \phi_{zz}] := \phi_{xx} + \phi_{yy} + \phi_{zz} - c^{-2}\phi_{tt} = 0$.

B.1 Generators

Consider the total space $\mathcal{F}$ characterized by $(t, \mathbf{x}, \phi)$, with projection map

$$\pi : \mathcal{F} \longrightarrow M \tag{B.1.1}$$

$$(t, \mathbf{x}, \phi) \longrightarrow \pi(t, \mathbf{x}, \phi) := (t, \mathbf{x}) . \tag{B.1.2}$$

Symmetry generators of equation (B.0.1) are vector fields in $T\mathcal{F}$, with

$$\mathcal{H} = \frac{1}{c}\frac{\partial}{\partial t} \tag{B.1.3}$$

$$\mathcal{P}_A = \frac{\partial}{\partial x^A} \tag{B.1.4}$$

$$\mathcal{J}_A = \epsilon_{ABC} x^B \frac{\partial}{\partial x^C} \tag{B.1.5}$$

$$\mathcal{K}_A = \frac{x^A}{c}\frac{\partial}{\partial t} + ct\frac{\partial}{\partial x^A} \tag{B.1.6}$$

$$S_0 = -2c^2 t x^A \frac{\partial}{\partial x^A} - \left(c^2t^2 + x^2 + y^2 + z^2\right)\frac{\partial}{\partial t} + 2c^2\phi\frac{\partial}{\partial \phi} \tag{B.1.7}$$

$$\mathcal{S}_A = 2x_A\left(x^B\frac{\partial}{\partial x^B} + t\frac{\partial}{\partial t}\right) - x^\mu x^\nu \eta_{\mu\nu}\frac{\partial}{\partial x^A} - 2x_A\frac{\partial}{\partial \phi} \tag{B.1.8}$$

$$\mathcal{D} = x^A\frac{\partial}{\partial x^A} + t\frac{\partial}{\partial t} \tag{B.1.9}$$

$$\mathcal{W} = \phi\frac{\partial}{\partial \phi}. \tag{B.1.10}$$

All symmetry generators except special conformal transformations leave the scalar field unchanged, which is the usual way one thinks of transformations of a scalar field.

B.2 Finite transformations

The finite symmetry transformations were obtained by solving systems of ODEs given by

$$\dot{\gamma}^{\mathcal{X}}(\lambda) = \mathcal{X}_{\gamma^{\mathcal{X}}(\lambda)}, \tag{B.2.1}$$

where $\gamma : \mathbb{R} \longrightarrow M$ is a curve in the base manifold M and $\mathcal{X} \in T\mathcal{F}$ is one of the symmetry generators. Solutions of these system of ODEs with initial conditions $\gamma^{\mathcal{X}}(0) = p$ for a given $p \in M$ are denoted by $\gamma_p^{\mathcal{X}}$. Each solution $\gamma_p^{\mathcal{X}}$ is unique and is used to construct a 1-parameter subgroup of the total group by building its associated flow

$$h^{\mathcal{X}} : \mathbb{R} \times \mathcal{F} \longrightarrow M \tag{B.2.2}$$

$$(\lambda, p) \longrightarrow h^{\mathcal{X}}(\lambda, p) := \gamma_p^{\mathcal{X}}(\lambda), \tag{B.2.3}$$

with transformations

$$h^{\mathcal{H}}(\lambda, t, \mathbf{x}, \phi) := (t + \lambda/c, \mathbf{x}, \phi) \tag{B.2.4}$$

$$h^{\mathcal{P}_A}(\lambda, t, \mathbf{x}, \phi) := (t, \mathbf{x} + \text{и}_A \lambda, \phi) \tag{B.2.5}$$

$$h^{\mathcal{J}_A}(\lambda, t, \mathbf{x}, \phi) := (t, R_A(\lambda)\, \mathbf{x}, \phi) \tag{B.2.6}$$

$$h^{\mathcal{D}}(\lambda, t, \mathbf{x}, \phi) := \left(e^{\lambda} t, e^{\lambda} \mathbf{x}, \phi\right) \tag{B.2.7}$$

$$h^{\mathcal{S}_\mu}(\lambda, x, \phi) := \left(\omega_\mu(\lambda) \left(x - \text{и}_\mu(\lambda) \langle x, x \rangle_\eta\right), \Omega_\mu(\lambda)\phi\right) \tag{B.2.8}$$

$$h^{\mathcal{W}}(\lambda, t, \mathbf{x}, \phi) := \left(t, \mathbf{x}, e^{\lambda}\phi\right), \tag{B.2.9}$$

rotation matrices given by

$$R_1(\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\lambda & -\sin\lambda \\ 0 & \sin\lambda & \cos\lambda \end{pmatrix} \qquad R_2(\lambda) = \begin{pmatrix} \cos\lambda & 0 & \sin\lambda \\ 0 & 1 & 0 \\ -\sin\lambda & 0 & \cos\lambda \end{pmatrix}, \tag{B.2.10}$$

and

$$R_3(\lambda) = \begin{pmatrix} \cos\lambda & -\sin\lambda & 0 \\ \sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{B.2.11}$$

indicator functions $и_0(\lambda) = (\lambda, 0, 0, 0)$, $и_1(\lambda) = (0, \lambda, 0, 0)$, $и_2(\lambda) = (0, 0, \lambda, 0)$, $и_3(\lambda) = (0, 0, 0, \lambda)$ and special conformal transformation related functions given by

$$\omega_\mu(\lambda) := \frac{\langle x, x\rangle_\eta}{\left\langle x - и_\mu(\lambda)\langle x, x\rangle_\eta, x - и_\mu(\lambda)\langle x, x\rangle_\eta \right\rangle_\eta} \quad \Omega_A(\lambda) := \langle \lambda x - и_A(1), \lambda x - и_A(1)\rangle_\eta, \tag{B.2.12}$$

and $\Omega_0(\lambda) := (c^2\lambda t + 1)^2 - c^2\lambda^2 x^2 - c^2\lambda^2 y^2 - c^2\lambda^2 z^2$.

B.2.1 Restriction to the space-time part

The restriction to the space-time component of these symmetries corresponds to the pushforward of the vector fields in the total space with respect to the projection map π, with

$$H = \pi_*\mathcal{H} = \frac{1}{c}\frac{\partial}{\partial t} \tag{B.2.13}$$

$$P_A = \pi_*\mathcal{P}_A = \frac{\partial}{\partial x^A} \tag{B.2.14}$$

$$J_A = \pi_*\mathcal{J}_A = \epsilon_{ABC} x^B \frac{\partial}{\partial x^C} \tag{B.2.15}$$

$$K_A = \pi_*\mathcal{K}_A = \frac{x^A}{c}\frac{\partial}{\partial t} + ct\frac{\partial}{\partial x^A} \tag{B.2.16}$$

$$D = \pi_*\mathcal{D} = x^A\frac{\partial}{\partial x^A} + t\frac{\partial}{\partial t} \tag{B.2.17}$$

$$S_A = \pi_*\mathcal{S}_A = 2x_A\left(x^B\frac{\partial}{\partial x^B} + t\frac{\partial}{\partial t}\right) - x^\mu x^\nu \eta_{\mu\nu}\frac{\partial}{\partial x^A}. \tag{B.2.18}$$

The group of symmetries is formed by flows h^X_λ with product being the map composition and $X = \pi_* \mathcal{X}$, which of course is also just $h_\lambda^X = \pi \circ h_\lambda^{\mathcal{X}}$.